CHAPTER 1

Exponential and Logarithmic Series

1. In the following chapter we are about to obtain an expansion in powers of x for the expression a^x, where both a and x are real, and also to obtain an expansion for $\log_e (1 + x)$, where x is real and less than unity, and e stands for a quantity to be defined.

2. *To find the value of the quantity* $\left(1+\frac{1}{n}\right)^n$, *when n becomes infinitely great and is real.*

Since $\frac{1}{n}<1,$ we have, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{1}{n^3}+...$$

$$=1+1+\frac{1-\frac{1}{n}}{1\cdot 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\underline{|4}}+... \qquad ...(1)$$

This series is true for all values of n, however great. Make then n infinite and the right-hand side

$$=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+... \text{ ad inf.} \qquad ...(2)$$

Hence the limiting value, when n is infinite, of $\left(1+\frac{1}{n}\right)^n$ is the sum of the series.

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}... \text{ ad inf.}$$

The sum of this series is always denoted by the quantity e.

Hence we have

$$\mathop{\mathbf{Lt}}_{\mathbf{n=\infty}}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{n}}\right)^{\mathbf{n}} = \mathbf{e},$$

where $\mathop{\text{Lt}}\limits_{n=\infty}$ stands for "the limit when $n = \infty$."

Cor. By putting $n = \frac{1}{m}$, it follows (since m is zero when n is infinity) that

$$\underset{m=0}{\text{Lt}}\ (1+m)^{1/m} = \underset{n=\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e.$$

3. This quantity e is finite.

For since $$\frac{1}{\lfloor 3} < \frac{1}{2 \cdot 2} < \frac{1}{2^2},$$

$$\frac{1}{\lfloor 4} < \frac{1}{2 \cdot 2 \cdot 2} < \frac{1}{2^3},$$

. . . .

we have

$$e < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \text{ ...ad inf.}$$

$$< 1 + \frac{1}{1 - \frac{1}{2}}$$

$< 1 + 2$ *i.e.* < 3.

Also clearly $e > 2$.

Hence it lies between 2 and 3.

By taking a sufficient number of terms in the series, it can be shown that

$e = 2.7182818285...$

4. *The quantity e is incommensurable.*

For, if possible, suppose it to be equal to a fraction $\frac{p}{q}$, where p and q are whole numbers.

We have then

$$\frac{p}{q} = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + ... + \frac{1}{\lfloor q} + \frac{1}{\lfloor q+1} + \frac{1}{\lfloor q+2} + ... \qquad ...(1)$$

Multiply this equation by $\lfloor q$, so that all the terms of the series (1) become integers except those commencing with $\frac{\lfloor q}{\lfloor q+1}$. Hence we have

$$p\lfloor q-1 = \text{whole number} + \frac{\lfloor q}{\lfloor q+1} + \frac{\lfloor q}{\lfloor q+2} + \frac{\lfloor q}{\lfloor q+3} + ...,$$

i.e. an integer $= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + ... \qquad ...(2)$

But the right-hand side of this equation is $> \frac{1}{q+1}$, and

$$< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \ldots,$$

i.e. $$< \frac{1}{q+1} \div \left(1 - \frac{1}{q+1}\right),$$

i.e. $$< \frac{1}{q}.$$

Hence the right-hand side of (2) lies between $\frac{1}{q+1}$ and $\frac{1}{q}$, and therefore a fraction and so cannot be equal to the left-hand side.

Hence our supposition that e was commensurable is incorrect and it therefore must be incommensurable.

5. Exponential Series: *When x is real, to prove that*

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots ad\ inf.,$$

and that

$$a^x = 1 + x \log_e a + \frac{x^2}{\lfloor 2}(\log_e a)^2 + \ldots\ ad\ inf.$$

When n is greater than unity, we have

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}^x = \left(1+\frac{1}{n}\right)^{nx}$$

$$= 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{1 \cdot 2}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{1 \cdot 2 \cdot 3}\frac{1}{n^3} + \ldots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1 \cdot 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1 \cdot 2 \cdot 3} + \ldots.$$

In this expression make n infinitely great. The left-hand becomes, as in Art. 2, e^x. The right-hand becomes

$$1 + x + \frac{x^3}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$$

Hence we have

$$\mathbf{e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ ad\ inf.} \qquad \ldots(1)$$

Let $a = e^c$, so that $c = \log_e a$.

$$\therefore \qquad a^x = e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \ldots \text{ ad inf.}$$

by substituting cx for x in the series (1).

$$\therefore \qquad \mathbf{a^x = 1 + x\log_e a + \frac{x^2}{\underline{|2}}(\log_e a)^2 + \frac{x^3}{\underline{|3}}(\log_e a)^3 + \ldots \text{ ad inf.}} \qquad \ldots(2)$$

6. It can be shown (as in C. Smith's *Algebra*, Art. 278) that the series (1), and therefore (2), of the last article is convergent for all real values of x.

7. Example 1. *Prove that* $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \ldots$ *ad inf.*

By equation (1) of Art. 5 we have, by putting x in succession equal to 1 and -1,

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots \text{ ad inf.}$$

and
$$e^{-1} = 1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \ldots \text{ ad inf.}$$

Hence, by subtraction,

$$e - e^{-1} = 2\left(1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \ldots\right),$$

i.e.
$$\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \ldots \text{ ad inf.}$$

Example 2. *Find the sum of the series*

$$1 + \frac{1+2}{\underline{|2}} + \frac{1+2+3}{\underline{|3}} + \frac{1+2+3+4}{\underline{|4}} + \ldots \text{ ad inf.}$$

The nth term
$$= \frac{1+2+3+\ldots+n}{\underline{|n}} = \frac{\frac{1}{2}n(n+1)}{\underline{|n}}$$

$$= \frac{1}{2}\frac{n+1}{\underline{|n-1}} = \frac{1}{2}\left[\frac{(n-1)+2}{\underline{|n-1}}\right] = \frac{1}{2}\left[\frac{1}{\underline{|n-2}} + \frac{2}{\underline{|n-1}}\right],$$

provided that $n > 2$.

Similarly,

$$\text{the } (n-1)\text{th term} = \frac{1}{2}\left[\frac{1}{\underline{|n-3}} + \frac{2}{\underline{|n-2}}\right],$$

. . . .

$$\text{the 4th term} = \frac{1}{2}\left[\frac{1}{\underline{|2}} + \frac{2}{\underline{|3}}\right],$$

$$\text{the 3rd term} = \frac{1}{2}\left[\frac{1}{\lfloor 1} + \frac{2}{\lfloor 2}\right].$$

Also $$\text{the 2nd term} = \frac{1}{2}\left[1 + \frac{2}{\lfloor 1}\right].$$

and $$\text{the 1st term} = \frac{1}{2}\left[\frac{2}{1}\right].$$

Hence, by addition, the whole series

$$= \frac{1}{2}\left[1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \text{ ad inf.}\right]$$

$$+ \frac{1}{2} \cdot 2\left[1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \text{ ad inf.}\right]$$

$$= \frac{1}{2} \cdot e + e = \frac{3e}{2}.$$

8. Logarithmic Series: *To prove that, when y is real and numerically* < 1, *then*

$$\log_e(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots \textit{ ad inf.}$$

In the equation (2) of Art. 5, put

$a = 1 + y$,

and we have

$$(1 + y)^x = 1 + x\log_e(1+y) + \frac{x^2}{\lfloor 2}\{\log_e(1+y)\}^2 + \ldots \qquad \ldots(1)$$

But, since y is real and numerically < unity, we have

$$(1 + y)^x = 1 + x \cdot y + \frac{x(x-1)}{1 \cdot 2}y^2 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}y^3 + \ldots \qquad \ldots(2)$$

The series on the right-hand side of (1) and (2) are equal to one another and both convergent, when y is numerically < 1. Also it could be shown that the series on the right-hand side of (2) is convergent when it is arranged in powers of x. Hence we may equate like powers of x.

Thus we have

$$\log_e(1+y) = y - \frac{y^2}{1 \cdot 2} + \frac{(-1)(-2)}{1 \cdot 2 \cdot 3}y^3 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3 \cdot 4}y^4 + \ldots \text{ad inf.},$$

i.e. $$\mathbf{\log_e(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 +} \text{ ad inf.} \qquad \ldots(3)$$

9. If $y = 1$, the series (3) of the previous article is equal to

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{... ad inf.}$$

which is known to be convergent.

If $y = -1$, it equals $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}$ ... ad inf. which is known to be divergent.

In addition therefore to being true for all values of y between -1 and $+1$, it is true for the value $y = 1$; it is not however true for the value $y = -1$.

10. Calculation of logarithms to base e.

In the logarithmic series, if we put $y = 1$, we have

$$\log_e 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{... ad inf.} \qquad \text{...(1)}$$

If we put $y=\frac{1}{2}$,

we have

$$\log_e 3-\log_e 2=\log_e \frac{3}{2}=\log_e\left(1+\frac{1}{2}\right)$$

$$=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\frac{1}{4}\cdot\frac{1}{2^4}+\ldots \qquad \text{...(2)}$$

If we put $y=\frac{1}{3}$.

we have

$$\log_e 4-\log_e 3=\log_e\left(1+\frac{1}{3}\right)=\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\frac{1}{4}\cdot\frac{1}{3^4}+\ldots \qquad \text{...(3)}$$

From these equations we could, by taking a sufficient number of terms, calculate $\log_e 2$, $\log_e 3$ and $\log_e 4$.

It would be found that a large number of terms would have to be taken to give the values of these logarithms to the required degree of accuracy. We shall therefore obtain more convenient series.

11. By Art. 8 we have

$$\log_e(1+y)\,y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\ldots \qquad \text{...(1)}$$

and, by changing the sign of y,

$$\log_e(1-y)=-y-\frac{1}{2}y^2-\frac{1}{3}y^3-\frac{1}{4}y^4+\ldots \qquad \text{...(2)}$$

In order that both these series may be true y must be numerically less than unity.

By subtraction, we have

$$\log_e(1+y)-\log_e(1-y)=\log_e\frac{1+y}{1-y}=2\left[y+\frac{1}{3}y^3+\frac{1}{5}y^5+\ldots\right] \quad \ldots(3)$$

Let $$y=\frac{m-n}{m+n},$$

where m and n are positive integers and $m > n$, so that

$$\frac{1+y}{1-y}=\frac{m}{n}.$$

The equation (3) becomes

$$\log_e\frac{m}{n}=2\left[\left(\frac{m-n}{m+n}\right)+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\ldots\right] \quad \ldots(4)$$

Put $m = 2$, $n = 1$ in (4) and we get $\log_e 2$.

Put $m = 3$, $n = 2$ and we get $\log_e 3 - \log_e 2$, and therefore $\log_e 3$.

By proceeding in this way we get the value of the logarithm of any number to base e.

12. Logarithms to base 10. The logarithms of the previous article, to base e, are called Napierian or natural logarithms.

We can convert these logarithms into logarithms to base 10.

For, by Art. 147 (Part I.), we have, if N be any number,

$$\log_e N = \log_{10} N \times \log_e 10.$$

$$\therefore \log_{10} N = \log_e N \times \frac{1}{\log_e 10}.$$

Now, $\log_e 10$ can be found as in the last article and then $\frac{1}{\log_e 10}$ is found to be 0.4342944819...,

Hence, $\log_{10} N = \log_e N \times 0.43429448...,$

so that the logarithm of any number to base 10 is found by multiplying its logarithm to base e by the quantity 0.43429448... This quantity is called the Modulus.

EXAMPLES I

Prove that

1. $\frac{1}{2}(e+e^{-1})=1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots$

2. $\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\right)\left(1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\ldots\right)=1.$

3. $\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\right)^2 = 1+\left(1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\ldots\right)^2 \ldots$

4. $1+\frac{2}{\underline{|3}}+\frac{3}{\underline{|5}}+\frac{4}{\underline{|7}}+\ldots=\frac{e}{2}.$

5. $\frac{2}{\underline{|3}}+\frac{4}{\underline{|5}}+\frac{6}{\underline{|7}}+\ldots=e^{-1}.$

6. $\dfrac{\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots}{1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\ldots}=\frac{e-1}{e+1}.$

7. $1+\frac{2^3}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^3}{\underline{|4}}+\ldots=5e.$

Find the sum of the series

8. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ ad inf.

9. $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\frac{1}{4}\cdot\frac{1}{2^4}+\ldots$ ad inf.

Prove that

10. $\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\ldots=\log_e a=-\log_e b.$

11. $\log_e\frac{1+x}{1-x}=2\left(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots\text{ad inf.}\right).$

12. $\log_e\frac{x+1}{x-1}=2\left(\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\ldots\text{ad inf.}\right)$, if $x>1$.

13. $\log_e(1+3x+2x^2)=3x-\frac{5x^2}{2}+\frac{9x^3}{3}-\frac{17x^4}{4}+\ldots+(-1)^{n-1}\frac{2^n+1}{n}x^n+\ldots,$

provided that $2x$ be not > 1.

14. $2\log_e x-\log_e(x+1)-\log_e(x-1)=\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\ldots,$ if $x>1$.

15. $\log_e 2=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\ldots$ ... ad inf.

16. $\log_e 2-\frac{1}{2}=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\ldots$ ... ad inf

17. $\tan\theta+\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta+\ldots=\frac{1}{2}\log\dfrac{\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)}$, if $\theta<\frac{\pi}{4}$.

18. If θ be > $\frac{\pi}{2}$ and < π, prove that

$$(1)\sin\theta+\frac{1}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta+\ldots\ldots\text{ ad inf.}$$

$$=2\left[\cot\frac{\theta}{2}+\frac{1}{3}\cot^3\frac{\theta}{2}+\frac{1}{5}\cot^5\frac{\theta}{2}+\ldots\text{ad inf.}\right],$$

and, if θ be > 0 and < $\frac{\pi}{2}$, prove that

$$(2)\frac{1}{2}\sin^2\theta+\frac{1}{4}\sin^4\theta+\frac{1}{6}\sin^6\theta+\ldots \text{ ... ad inf.}$$

$$=2\left[\tan^2\frac{\theta}{2}+\frac{1}{3}\tan^6\frac{\theta}{2}+\frac{1}{5}\tan^{10}\frac{\theta}{2}+\ldots \text{ ad inf.}\right]$$

19. If $\tan^2\theta < 1$, prove that

$$\tan^2\theta-\frac{1}{2}\tan^4\theta+\frac{1}{3}\tan^6\theta-\ldots \text{ ad inf.}$$

$$=\sin^2\theta+\frac{1}{2}\sin^4\theta+\frac{1}{3}\sin^6\theta+\ldots \text{ ad inf.}$$

20. Prove that, if 2θ be not a multiple of π,

$$\log\cot\theta=\cos 2\theta+\frac{1}{3}\cos^3 2\theta+\frac{1}{5}\cos^5 2\theta+\ldots \text{ ad inf.}$$

21. Prove that the coefficient of x^n in the expansion of

$$\{\log_e(1+x)\}^2$$

is

$$\frac{2(-1)^n}{n}\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right].$$

22. Use the methods of Arts. 11 and 12 to prove that

$$\log_{10} 2 = 0.30103\ldots$$

and

$$\log_{10} 3 = 0.47712\ldots$$

23. Draw the curve $y = \log_e x$.

[If x be negative, y is imaginary; when x is zero, y equals $-\infty$; when x is the unity, y is nothing; when x is positive and > 1, y is always positive; when x is infinity, y is infinity also.]

24. Draw the curve $y = \log_{10} x$ and state the geometrical relation between it and the curve of the last example.

[Use Art. 147, Part I.]

25. Draw the curve $y = a^x$.

13. The two following limits will be required in the next chapter but one.

14. *To prove that the value of* $\left(\cos\frac{a}{n}\right)^n$, *when n is infinite, is unity.*

We have
$$\cos\frac{a}{n}=\left(1-\sin^2\frac{a}{n}\right)^{\frac{1}{2}}.$$

$$\therefore \quad \left(\cos\frac{a}{n}\right)^n=\left(1-\sin^2\frac{a}{n}\right)^{\frac{n}{2}}=\left[\left(1-\sin^2\frac{a}{n}\right)^{-\frac{1}{\sin^2\frac{a}{n}}}\right]^{-\frac{n}{2}\sin^2\frac{a}{n}}.$$

Now, by putting

$$-\sin^2\frac{a}{n}=m,$$

We have

$$\underset{n=\infty}{\text{Lt}}\left\{1-\sin^2\frac{a}{n}\right\}^{-\frac{1}{\sin^2\frac{a}{n}}}=\underset{m=0}{\text{Lt}}\{1-m\}^{\frac{1}{m}}=e. \qquad \text{(Art. 2, Cor.)}$$

Also, by Art. 228 (Part I.),

$$\frac{n}{2}\sin^2\frac{a}{n}$$

$$=\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)^2\times\frac{a^2}{2n}=1\times 0=0,$$

when n is infinite.

Hence, when n is infinite,

$$\left[\cos\frac{a}{n}\right]^n=e^0=1.$$

Alter. This limit may also be found by using the logarithmic series.

For, putting $\left(\cos\frac{a}{n}\right)^n=u,$ we have

$$\log_e u=n\log_e\cos\frac{a}{n}=\frac{n}{2}\log_e\cos^2\frac{a}{n}$$

$$=\frac{n}{2}\log_e\left(1-\sin^2\frac{a}{n}\right)$$

$$=-\frac{n}{2}\left(\sin^2\frac{a}{n}+\frac{1}{2}\sin^4\frac{a}{n}+\frac{1}{3}\sin^6\frac{a}{n}+\ldots\right).$$

(Art. 8.)

The series inside the bracket lies between $\sin^2\frac{a}{n}$ and the series

$$\sin^2\frac{a}{n}+\sin^4\frac{a}{n}+\sin^6\frac{a}{n}+\ldots\text{ ad inf.},$$

i.e. lies between

$$\sin^2\frac{a}{n}\text{ and }\frac{\sin^2\frac{a}{n}}{1-\sin^2\frac{a}{n}},$$