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Coaxial circles

The radical axis of two circles

A quadratic equation of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

represents a circle, center $(-g, -f)$ and radius the square root of $g^2 + f^2 - c$. It is imaginary when this latter quantity is negative. Every circle equation can be normalized so that the coefficients of x^2 and y^2 are both 1 (and there is no xy term). We call this the standard equation of a circle.

The power of a point P with respect to a circle $C(O, r)$ is the quantity $OP^2 - r^2$. If P is external to $C(O, r)$, and T a point on the circle such that PT is tangent to the circle, then the power of P with respect to C is the squared length of PT .

Lemma 1. *Let C be the circle with standard equation*

$$f(x, y) := x^2 + y^2 + 2gx + 2fy + c = 0.$$

For every point $P = (u, v)$, $f(u, v)$ is the power of P with respect to the circle.

The radical axis of two circles is the locus of points with equal powers with respect to the two circles. The radical axis of the two circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is clearly the straight line

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0.$$

Construction

If two circles intersect, their radical axis is clearly the line containing their intersections. In case of tangency, it is the common tangent at the intersection. Here is a construction of the radical axis in the general case.

Coaxial circles

Proposition 2. *Given two circles $C(O_1, r_1)$ and $C(O_2, r_2)$, let P_1, P_2 be points on the respective circles and on the same side of the line O_1O_2 , such that O_1P_1 and O_2P_2 are perpendicular to O_1O_2 . If the perpendicular bisector of P_1P_2 intersects O_1O_2 at Q , and Q' is the point on O_1O_2 such that $O_1Q' = QO_2$ and $O_2Q' = QO_1$, then the perpendicular to O_1O_2 at Q' is the radical axis of the two circles.*

$O_1 Q' Q O_2$

P_1

P_2

Exercise

1. Prove that the radical axis of two nonconcentric circles is perpendicular to the line joining the centers of the circles.

The radical center of three circles

Consider three circles

$$x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0,$$

$$x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0,$$

$$x^2 + y^2 - 2a_3x - 2b_3y + c_3 = 0.$$

The radical axes of the three pairs of circles being

$$(a_2 - a_3)x + (b_2 - b_3)y - (c_2 - c_3) = 0,$$

If the centers of the circles are not on a line, the three radical axes intersect at a point, which has equal powers with respect to the three circles. This is called the *radical center* of the three circles.

Example 3.1. The radical center of the excircles of a triangle.

The radical axis of the excircles (I_b) and (I_c) is a line perpendicular to I_bI_c . It is parallel to the bisector of angle A, and clearly contains the midpoint D of BC. It follows that the radical axis is the D-bisector of the medial triangle DEF; similarly for the other two radical axes. We conclude that the radical center of the excircles of ABC is the incenter of the medial triangle DEF. This is called the *Spijker center* of triangle ABC.

Orthogonal circles

Two circles are orthogonal if they intersect at right angles (at each intersection).

Proposition 3.3. *The two circles with standard equations*

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

are orthogonal if and only if

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 0.$$

Proof. The orthogonality condition is

$$0 = (r_2$$

$$+ r_2$$

$$)^2 - (g_1 - g_2)^2 - (f_1 - f_2)^2$$

$$= (g_2$$

$$+ f_2$$

$$)^2 - c_1 + (g_2$$

$$+ f_2$$

$$)^2 - c_2 - (g_1 - g_2)^2 - (f_1 - f_2)^2$$

$$= 2g_1g_2 + 2f_1f_2 - c_1 - c_2.$$

Exercise

1. Construct a circle orthogonal to three given circles whose centers are not collinear.

2. Given three points A, B, C which form the vertices of an acute angled triangle, construct three mutually orthogonal circles with centers A, B, C.

Consider the possibility of a circle orthogonal to two given nonconcentric circles. We may assume the radical axis of the two given circles to be the y-axis. The standard equations of the circles are

$$x^2 + y^2 + 2g_1x + c_0 = 0,$$

$$x^2 + y^2 + 2g_2x + c_0 = 0.$$

If there is a circle with standard equation $x^2 + y^2 + 2gx + 2fy + c = 0$ orthogonal to both of these circles, then

$$2g_1g - c_0 - c = 0,$$

$$2g_2g - c_0 - c = 0.$$

Since $g_1 \neq g_2$, we must have $g = 0$ and $c = -c_0$.

(1) If the circles intersect on the y-axis at $(0, \pm b)$, then $c_0 = -b^2$, then the common orthogonal circle is $x^2 + y^2 + 2fy + b^2 = 0$ for some f satisfying $f^2 \geq b^2$.

(2) If the circles do not intersect, c_0 must be positive, say, $c_0 = b^2$. In this case, the

common orthogonal circle is $x^2 + y^2 + 2fy - b^2 = 0$. This always passes through the points $(-b, 0)$.

Coaxial circles

A system of circles is coaxial if every pair of circles from the system have the same radical axis. Therefore, a coaxial system of circles is defined by the radical axis and any one of the circles.

Example 2. Let c be a constant and λ a parameter. The equation $x^2 + y^2 + 2\lambda x + c = 0$ defines a coaxial system of circles with centers on the x -axis. The common radical axis is the y -axis, the line $x = 0$.

(i) The circles cut the radical axis if $c < 0$. In this case, the common points of the circles in the systems are $(0, \pm\sqrt{-c})$.

(ii) If $c = 0$, the circles are all tangent to the y -axis.

(iii) If $c > 0$, the circles do not have common points. For $\lambda = \pm\sqrt{c}$, the two circles degenerate into two points, called the limiting points of the coaxial system.

Example 3 Let $c \neq 0$ be a fixed number. The two systems of coaxial circles

$$x^2 + y^2 + 2\lambda y + c = 0,$$

$$x^2 + y^2 + 2\mu x + c = 0$$

are every circle in one system is orthogonal to every circle in the other system. These two systems of coaxial circles are said to be conjugate.

Exercise

1. Let c be a constant. Show that every circle from the coaxial system $x^2 + y^2 + 2\lambda x + c = 0$ is orthogonal to every circle in the coaxial system $x^2 + y^2 + 2\mu y - c = 0$.

2. A and B are two fixed points, and P is a variable point such that $AP = k \cdot BP$.

(a) Show that the locus of P is a circle.

(b) Show that for different values of k , the circles form a coaxial system with A and B as limiting points.

Miscellaneous exercises

1. Prove that, for all values of the constants p and q , the circle

$$(x - a)(x - a + p) + (y - b)(y - b + q) = r^2$$

bisects the circumference of the circle

$$(x - a)^2 + (y - b)^2 = r^2.$$

Make use of this to find the equation of the circle which bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$ and touches the line $x - y = 0$ at the origin.

2. Find the equation of the circle which is orthogonal to all of the circles

$$x^2 + y^2 + 3x + 5y = 0,$$

$$x^2 + y^2 + 2x + 4y + 3 = 0,$$

$$x^2 + y^2 + 4x + 5y - 1 = 0.$$

3. Given a circle C with center O and a line ℓ , let A be the pedal of O on ℓ . For a variable point P on C , let the circle with diameter AP intersect the C (again) at X and ℓ (again) at Y . Show that the line XY passes through a fixed point as P varies on C .

4. Let C_1 and C_2 be two circles on the same side of ℓ , none of them intersecting ℓ . Locate the point Q on ℓ for which the sum of the lengths of the tangents from Q to the circles is minimum.

5. Let P be a fixed point outside the circle $C(0, r)$. A line ℓ not containing P intersects

the circle at two points X and Y such that $PX \cdot PY = OP^2 - r^2$. Show that ℓ passes through a fixed point.