VIKAS KUMAR
C.M.J.College,Khutauna
Mathematics

System of Circles

1. A set of circles is said to be a ***system of circles*** if it contains atleast two circles.

2. Two circles $S = 0$ and $S' = 0$ are said to *touch each other* if they have if they have a unique point P in common. The common point P is called ***point of contact*** of the circles $S = 0$ and $S' = 0$.

3. If two circles touch each other then there exists only one tangent at the point of contact of the two circles.

4. Let $S = 0$, $S' = 0$ be two circles with centres c_1, c_2 and radii r_1 , r_2 respectively.

5. If $C_1C_2 > r_1 + r_2$ then each circle lies completely outside the other circle.

6. If $C_1C_2 = r_1 + r_2$ then the two circles touch each other externally. The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ internally.

7. If $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ then the two circles intersect at two points P and Q. The chord PQ is called common chord of the circles.

8. If $C_1C_2 = |r_1 - r_2|$ then the two circles touch each other internally. The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ externally.

9. If $C_1C_2 < |r_1 - r_2|$ then one circle lies completely inside the other circle.

10. If two circles $S = 0$ and $S' = 0$ intersect at P, then the angle between the tangents of the two circles at P is called the ***angle between the circles*** at P.

11. If d is the distance between the centres of two intersecting circles with radii r_1, r_2 and θ is the angle between the circles then $\cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$

12. If θ is the angle between the circles $S = x^2 + y^2 + 2gx + 2fy + c = 0$, $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ then $\cos\theta = \dfrac{c + c' - 2(gg' + ff')}{2\sqrt{g^2 + f^2 - c}\,\sqrt{g'^2 + f'^2 - c'}}$

13. Two intersecting circles are said to cut each other ***orthogonally*** if the angle between the circles is a right angle.

14. Let d be the distance between the centres of two intersecting circles with radii r_1, r_2. The two circles cut orthogonally iff $d^2 = r_1^2 + r_2^2$.

15. The condition that the two circles $S \equiv x^2+y^2 + 2gx + 2fy + c = 0$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$
may cut each other orthogonally is $2gg' + 2ff' = c + c'$.
16. A common tangent $L = 0$ of the circles $S = 0$, $S' = 0$ is said to be a ***direct common tangent*** of the
circles if the two circles $S = 0$. $S' = 0$ lie on the same side of $L = 0$.
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17. A common tangent $L = 0$ of the circle $S = 0$, $S' = 0$ is said to be a ***transverse common tangent*** of
the circles if the two circles $S = 0$, $S' = 0$ lie on the opposite (either) sides of $L = 0$.
18. Let $S = 0$, $S' = 0$ be two circles with centres C_1, C_2 and radii r_1, r_2 respectively and n be the number of common tangents.
19. If $C_1C_2 > r_1 + r_2$ then $n = 4$
20. If $C_1C_2 = r_1 + r_2$ then $n = 3$
21. If $|r_1 - r_2| < C_1 C_2 < r_1 + r_2$ then $n = 2$
22. If $C_1C_2 = | r_1 - r_2|$ then $n = 1$
23. If $C_1C_2 < |r_1 - r_2|$ then $n = 0$
24. Let $S = 0$, $S' = 0$ be two circles. (i) The point of intersection of direct common tangents of $S = 0$,
$S' = 0$ is called ***external centre of similitude.*** (ii) The point of intersection of transverse common
tangents $S = 0$, $S' = 0$ is called ***internal centre of similitude.***
25. Let $S = 0$, $S' = 0$ be two circles with centres C_1, C_2 and radii r_1, r_2 respectively. If A_1 and A_2 are
respectively the internal and external centres of similitude of the circles $S = 0$, $S' = 0$ then
i) A_1 divides C_1C_2 in the ratio $r_1 : r_2$ internally
ii) A_2 divides C_1C_2 in the ratio $r_1 : r_2$ externally
26. If the radii of two circles are equal then the external centre of similitude does not exist.
27. The locus of a point, for which the powers with respect of two given nonconcentric circles are equal, is a straight line, called the ***radical axis*** of the given circles.
28. The equation of the radical axis of the circles $S= 0$, $S' = 0$ is $S - S' = 0$.
29. The lengths of tangent from a point on the radical axis of two circles are equal, if exists.
30. The radical axis of two circles bisects all common tangents of the two circles.
31. The radical axis of two circles is perpendicular to their line of centres.
32. If two circles intersect, then the radical axis is their common chord.
33. If two circles touch each other, then the radical axis is their common tangent at the point of contact.
34. Any point on the radical axis of two circles $S=0$, $S' = 0$ lies externally or lies internally or lies on
both the circles simultaneously.
35. The radical axes of three circles, whose centres are noncollinear, taken in pairs, are concurrent.
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36. The point of concurrence of the radical axes of three circles, whose centres are noncollinear, taken in pairs, is called the ***radical centre*** of the circles.
37. The powers of the radical centre of three circles with respect to each of the three circles are equal.
38. The centre of a circle cutting two circles orthogonally lies on the radical axis of the two circles.

39. The centre of the circle cutting three circles orthogonally is the radical centre of the three circles.
The radius of the circle cutting three circles orthogonally is the length of tangent from the radical centre to any of the three circles.
40. If P is the radical centre of three circles and r is the length of tangent from P to any of the circles
then the circle with centre P and radius r cuts the three circles orthogonally.
41. A system of circles is said to be a ***system of coaxal circles*** or ***coaxal system of circles*** if every pair of circles has the same radical axis.
42. Since the radical axis is perpendicular to the line of centres, it follows that the centres of circles in
a coaxal system are collinear.
43. If $S=0$, $S' = 0$ are two circles then $\lambda_1 S + \lambda_2 S' = 0$ where λ_1, λ_2 are parameters such that $\lambda_1 + \lambda_2 \neq 0$,
represents the coaxal system of all circles containing $S = 0$, $S' = 0$.
44. If $S = 0$, $S' = 0$ are two circles then the coaxal system $\lambda_1 S + \lambda_2 S' = 0$ is called the coaxal system
determined by the circles $S = 0$, $S' = 0$.
45. If two circles intersect, then the radical axis is the common chord and hence $\lambda_1 S + \lambda_2 S' = 0$
represents a coaxal system of circles passing through the points of intersection of the circles $S = 0$, $S' = 0$.
46. If $S = 0$ is a circle and $L = 0$ is a line then $S + \lambda L = 0$ where λ is a parameter, represents the coaxal
system of all circles of which $S = 0$ is a member and $L = 0$ is the radical axis of the system.
47. The coaxal system $S + \lambda L = 0$ is called the coaxal system determined by the circle $S = 0$ and the
line $L = 0$ as the radical axis.
48. If $S = 0$ is a circle in the coaxal system having radical axis $L = 0$ then every circle in the system is
of the form $S + \lambda L = 0$ for some constant λ.
49. If $S = 0$, $S' = 0$ be two circles then every circle in the coaxal system $\lambda_1 S + \lambda_2 S' = 0$ except $S' = 0$
can be taken as $S + \lambda L = 0$ for some constant λ where $L = S - S'$.
50. The coaxal system of circles is said to be in the ***simplest form*** it its line of centres is x-axis and
the radical axis is y –axis.
51. The equation to the system of coaxal circles in the simplest form is $x^2 + y^2 + 2\lambda x + c = 0$ where λ
is a parameter and c is a fixed constant.
52. Let $x^2 + y^2 + 2\lambda x + c = 0$, λ is a parameter, c is a fixed constant, be a coaxal system of circles.
Then
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i) If $c < 0$ then the system of circles is an intersecting coaxal system.
ii) If $c = 0$ the then system of circles is a touching coaxal system.
iii) If $c > 0$ then the system of circles is a nonintersecting coaxal system.
53. The point circles in a coaxal system are called the ***limiting points*** of the coaxal system.
54. The limiting points of the coaxal system $x^2 + y^2 + 2\lambda x + c = 0$ are $(\mp c, 0)$.
55. A nonintersecting coaxal system has two limiting points.

56. A touching coaxal system has only one limiting point which is the point of contact of the circles of the system.

57. An intersecting coaxal system has no limiting point.

58. If P and Q are the two limiting points of a coaxal system then Q is the image of P with respect to the radical axis.

59. If P, Q are the two limiting points of a coaxal system then P, Q are inverse points with respect to the coaxal system.

60. The limiting points of a coaxal system are conjugate points with respect to every circle in the coaxal system.

61. The coaxal system of circle $\rho = 0$ is said to be ***orthogonal*** to a coaxal system of circles $\rho' = 0$ if every circle in $\rho = 0$ is orthogonal to every circle in $\rho' = 0$.

62. The equation of the coaxal system of circles orthogonal to the coaxal system $x^2 + y^2 + 2\lambda x + c = 0$ is $x^2 + y^2 + 2\mu y - c = 0$ where μ is a parameter.

63. The coaxal systems $\rho \equiv x^2 + y^2 + 2\lambda x + c = 0$, $\rho' = x^2 + y^2 + 2\mu y - c = 0$ are called ***conjugate coaxal systems.***

64. If $\rho = x^2 + y^2 + 2\lambda x + c = 0$ is a nonintersecting coaxal system of circles then every circle in the coaxal system $\rho' = x^2 + y^2 + 2\mu y - c = 0$ passes through the limiting points of $\rho = 0$.

65. If $r_1, r_2 (r_1 > r_2)$ are the radii of two circles and d is the distance between the centres of the circles then

i) the length of the direct common tangent = $\sqrt{d^2 - (r_1 - r_2)^2}$.

ii) the length of the transverse common tangent = $\sqrt{d^2 - (r_1 + r_2)^2}$.

66. The line joining the midpoints of the pair of tangents drawn from P to the circle S = 0 is the radical axes of the point circle of P and S = 0.

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67. If origin is a limiting point of the coaxal system containing the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then the other limiting point is $\left(\frac{-gc}{g^2+f^2}, \frac{-fc}{g^2+f^2}\right)$.

68. A common tangent drawn to any two circles of a coaxial system subtends angles at the limiting

points which are equal to $\pi/2$.

69. A circle passing through the limiting points of a given co-axal system cuts any member of the system at an angle of 90°

70. Length of direct common tangent is $\sqrt{(c_1c_2)^2 - (r_1 - r_2)^2}$. Where c_1, c_2 are centres and r_1, r_2 are radii of given circles.

71. Length of Transverse common tangent is $\sqrt{(c_1c_2)^2 - (r_1 + r_2)^2}$.

72. Two circles of radii r_1 and r_2 intersect at angle θ. The length of their common chord is

$$\frac{2r_1r_2\sin\theta}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}}.$$

73. Two circles with centres C_1 and C_2 and radius a cut each other orthogonally. Then $a = \frac{C_1C_2}{\sqrt{2}}$