Lesson 10

SUCCESSIVE DIFFERENTIATION: LEIBNITZ'S THEOREM

OBJECTIVES

At the end of this session, you will be able to understand:

- Definition
- n^{th} Differential Coefficient of Standard Functions
- Leibnitz's Theorem

DIFFERENTIATION: If $y = f(x)$ be a differentiable function of x, then $\frac{dy}{dx} = f'(x)$ is called the first differential coefficient of y w.r.t x.

Hence, differentiating both side w.r.t. x, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{d}{dx}\right)\left[f'(x)\right] = f''(x).$$

Let $\left(\frac{d}{dx}\right)\left(\frac{dy}{dx}\right)$ be represented by $\frac{d^2y}{dx^2}$; then $\frac{d^2y}{dx^2} = f''(x)$

Similarly $\left(\frac{d}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$ is represented by $\frac{d^3y}{dx^3}$; ie $\frac{d^3y}{dx^3} = f'''(x)$ and so on

The expressions $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{d^3x}, \ldots\ldots\ldots \frac{d^ny}{dx^n}$ are called the first, second, third, ...nth differential coefficient of y.

These function are usually written as

$y', y'', y''', \ldots\ldots y^n$ or $y_1, y_2 y_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots y_n$ and also $Dy, D^2y, D^3y, \ldots\ldots\ldots D^ny$.

n^{th} DIFFERENTIAL COEFFICIENT OF STANDARD FUNCTION:

(i) Differential Coefficient of x^m :

If $y = x^m$, then $y_1 = mx^{m-1}$;

$y_2 = m(m-1)x^{m-2}; y_3 = m(m-1)(m-2)x^{m-3}$ and so on

In general, $y_n = m(m-1)(m-2)(m-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(m-n+1)x^{m-n}$;

Note: If m be a positive integer, we have

$$y_n = 1.2.3............m = m!;$$

Hence $D_n(x^m) = m(m-1)(m-2)(m-3)............(m-n+1)x^{m-n}$

(ii) Differential Coefficient of $(ax+b)^m$:

If y = $(ax+b)^m$, then $y_1 = am(ax+b)^{m-1}$;

$y_2 = a^2 m(m-1)(ax+b)x^{m-2}$; $y_3 = a^3 m(m-1)(m-2)(ax+b)x^{m-3}$ and so on.

In general $y_n = a^n m(m-1)(m-2)(m-3)............(m-n+1)(ax+b)x^{m-n}$

[Note: In the first differentiation, the last term in it is $(m-1+1)$; in the second differentiation it is $(m-1)$ i.e. $(m-2+1)$;in the third differentiation it is $(m-2)$ i.e. $(m-3+1)$.So the nth differentiation it will be $(m-n+1)$.]

Hence $D(ax+b)^m = a^n m(m-1)(m-2)(m-3)............(m-n+1)(ax+b)^{m-n}$

or $D(ax+b)^m = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$

In case m is negative integer,let $m-p$, where p is positive integer, then

$D(ax+b)^{-p} = a^n(-p)(-p-1)(-p-2)....................[-p-(n-1)](ax+b)^{-p-n}$

$= a^n(-1)^n p(p+1)(p+2)....................[(p-n+1)](ax+b)^{-p-n}$

$= (-1)^n \frac{(p-n+1)!}{(p-1)!} a^n (ax+b)^{-p-n}$

Note1. If $m = n$.then $D^n(ax+b)^{-p} = a^n!$

Note2. If $m = -1$, we have $D^n(ax+b)^{-1} = (-1)(-1-1)................(-1-n+1)a^n(ax+b)^{-1-n}$

$(-1)^n 1.2.3...........na^n(ax+b)^{-1-n} = (-1)^n n! a^n (ax+b)^{-1-n} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.

(iii) Differential Coefficient of $(ax+b)$:

If $y = \log(ax+b), then$ $\quad y_1 = \frac{a}{(ax+b)} = a(ax+b)^{-1} = \frac{a(0!)}{(ax+b)}$

$$y_2 = \frac{a^2.1}{(ax+b)^2} = -\frac{a^2.(1!)}{(ax+b)^2}.y_3 = \frac{a^3.2}{(ax+b)^3} = -\frac{a^3.(2!)}{(ax+b)^3}.$$

$$y_4 = \frac{a^4 2.3}{(ax+b)^4} = (-1)^3.\frac{a^4.(3!)}{(ax+b)^4}$$ and so on.

In general, $y_n = (-1)^{n-1}.\dfrac{a^n.(n-1)!}{(ax+b)^n}$

Hence $D^n \log(ax+b) = (-1)^{n-1}.\dfrac{a^n.(n-1)!}{(ax+b)^n}.$

Note: $D^n \log x = \dfrac{(-1)^{n-1}(n-1)!}{x^n}.$

(iv) Differential Coefficient of a^{bx} **:**

If $y = a^{bx}$, then $y_1 = ba^{bx}\log a, y_2 = b^2 a^{bx}(\log a)^2.$

$y_3 = b^3 a^{bx}(\log a)^3$, and so no.

In general, $y_n = b^n a^{bx}(\log a)^n$,

Hence $\mathrm{D}^{\mathrm{n}}.a^{bx} = b^n a^{bx}(\log_e a)^n.$

(v) Differential Coefficient of e^{ax} **:**

If $y = e^{ax}$, then

$y_1 = ae^{ax}, y_2 = a^2 e^{ax}, y_3 = a^3 e^{ax}, y_4 = a^4 e^{ax}$ and so on.

In general, $y_n = a^n e^{ax}$, Hence $D^n e^{ax} = a^n e^{ax}.$

(vi) Differential Coefficient of $\sin(ax+b)$ **:**

If $y = \sin(ax+b)$, then

$$y_1 = a\cos(ax+b) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

$$y_2 = -a^2\cos(ax+b) = a^2\sin\left[\frac{2\pi}{2} + (ax+b)\right], \text{and so on.}$$

In General, $y_1 = a^n \sin\left(ax+b+\frac{1}{2}n\pi\right).$

Hence, $D^n(ax+b) = a^n \sin\left[ax+b+\frac{1}{2}n\pi\right]$

Note: $D^n \sin x = \sin\left[x + \left(\frac{n\pi}{2}\right)\right]$

(vii) Differential Coefficient of $\cos(ax+b)$ **:**

If $y=\cos(ax+b)$, then

$y_1=-a\sin(ax+b)=a\cos\left(\frac{\pi}{2}+ax+b\right)$

$y_2=-a^2\sin\left(\frac{\pi}{2}+ax+b\right)=a^2\cos\left(\frac{2\pi}{2}+ax+b\right)$, and so on

In general, $y_n=a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$.

Hence, $D^n\cos x(ax+b)=a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$.

Note : $D^n\cos x=\cos\left(x+\frac{1}{2}n\pi\right)$.

(viii) Differential Coefficient of $e^{ax}\sin(bx+c)$ and $e^{ax}\cos(bx+c)$ **:**

If $y=e^{ax}\sin(bx+c)$, then

$y_1=e^{ax}\sin(bx+c)+be^{ax}\cos(bx+c)=e^{ax}[a\sin(bx+c)+b\cos(bx+c)]$

Putting $a=r\cos\phi$ and $b=r\sin\phi$, we get

$y_1=re^{ax}\sin(bx+c+\phi)$, where $r^2=a^2+b^2$ and $\phi=\tan^{-1}\left(\frac{b}{a}\right)$,

similary, $y_1=r^2e^{ax}\sin(bx+c+2\phi)$, and so on.

Hence, $D^ne^{ax}\sin(bx+c)=r^ne^{ax}\sin(bx+c+n\phi)$

where $r=(a^2+b^2)^{\frac{1}{2}}$ and $\phi=\tan^{-1}\left(\frac{b}{a}\right)$.

Similarly, $D^ne^{ax}\cos(bx+c)=r^ne^{ax}\cos(bx+c+n\phi)$

where $r=(a^2+b^2)^{\frac{1}{2}}$ and $\phi=\tan^{-1}\left(\frac{b}{a}\right)$.

Example. Find the n^{th} derivative of $e^{ax}\sin bx\cos cx$.

Solution.

Let $y = e^{ax} \sin bx \cos cx = \frac{1}{2} e^{ax} (2 \sin bx \cos cx)$

$$= \frac{1}{2} e^{ax} [\sin(bx + cx) + \sin(bx - cx)] = \frac{1}{2} [e^{ax} \sin(b+c)x + e^{ax} \sin(b-c)x].$$

Now $D^n \left[e^{ax} \sin(bx + cx) \right] = (b^2 + c^2)^{n/2} e^{ax} \sin\left[bx + c + n \tan^{-1}\left(\frac{b}{a}\right)\right]$

$$\therefore y^n = \frac{1}{2} \left[\begin{array}{l} \left\{a^2 + (b+c)^2\right\}^{n/2} e^{ax} \sin\left\{(b+c)x + n \tan^{-1}(b+c)/a\right\} \\ + \left\{a^2 (b-c)^2\right\}^{n/2} e^{ax} \sin\left\{(b-c)x + n \tan^{-1} \frac{(b-c)}{a}\right\} \end{array} \right]$$

Example. If $y = e^{ax} \sin bx$, proved that $y_2 - 2ay_1 + (a^2 + b^2) y = 0.$

Solution.

Let $y = e^{ax} \sin bx,$ $\qquad \therefore\ y_1 = ae^{ax} \sin bx + be^{ax} \cos bx$

and $y_2 = a^2 e^{ax} \sin bx + 2abe^{ax} - b^2 e^{ax} \sin bx$ ....(1)

Also $-2ay_1 = -2a^2 e^{ax} \sin bx - 2abe^{ax} \cos bx$ ....(2)

and $(a^2 + b^2) y = a^2 e^{ax} \sin bx + b^2 e^{ax} \sin bx$ ....(3)

Adding (1),(2)and (3), we get

$$y_2 - 2ay_1 + (a^2 + b^2) y = 0$$

LEIBNITZ'S THEOREM:

The find n[th] differential coefficient of two function of x

If u and v are any two functions of x such that all their desired differential coefficients exist, then the n[th] differential coefficient of their product is given by

$$D^n (uv) = (D^n u).v + {}^n c_1 D^{n-1} u.Dv + {}^n c_2 D^{n-1} u.D^2 v + ... + {}^n c_r D^{n-r} n.D^r v + ... + uD^n v.$$

or

$$D^n (uv) = (D^n u).v + nD^{n-1} u.Dv + \frac{n(n-1)}{2!} D^{n-2} uD^2 v + ... + nDuD^{n-1} v + uDv.$$

Proof.

Let $y = uv$, we have

$$Dy(uv) = (D^n u) = (Du).v + u.Dv \qquad \text{........ ......(1)}$$

From (1) we see that the theorem is true for n $= 1$.

Now assume that the theorem is true for a particular value of n,we have

$$D^n(uv) = (D^n u).v + n_{c1} D^{n-1}u.Dv + n_{c2} D^{n-2}u.D^2 v + ... +^n c_r D^{n-r}u.D^r v$$
$$+ n_{r+1} D^{n-r-1} u D^{r+1} v + ... + uD^n v \qquad \text{............(2)}$$

Differentiating both siede of (2) w,r,t,x, we get

$$D^{n+1}(uv) = \left[(D^{n+1}u).v + D^n u Dv\right] + \left(n_{c1} D^n u.Dv + n_{c1} D^{n-1}u.D^2 v\right)$$
$$+ \left(n_{c2} D^{n-1}u.D^2 v + n_{c2} D^{n-2}u.D^3 v\right) + ... + \left(n_{cr} D^{n-r+1}u.D^r v + n_{cr} D^{n-r}u.D^{r+1} v\right)$$
$$+ \left(n_{cr+1} D^{n-r}u.D^{r+1} v + n_{cr+1} D^{n-r-1}u.D^{r+2} v\right) + ... + \left(DuD^n v + u.D^{r+1} v\right).$$

Rearranging the term, we get

$$D^{n+1}(uv) = (D^{n+1}u).v + (1 + n_{c1}) + (D^n u Dv) + (n_{c1} + n_{c2}) D^{n-1}u.D^2 v +$$
$$..... + (n_{cr} + n_{r+1})(D^{n-r}u.D^{r+1} v) + ... + uD^{n+1} v \qquad \text{...(3)}$$

But we know that ${}^n c_1 +^n c_{r+1} =^{n+1} c_{r+1}$. Therefore

$1 +^n c_1 =^n c_0 +^n c_1 =^{n+1} c_1$, ${}^n c_1 +^n c_2 =^{n+1} c_{2,}$ and so on.

Hence(3) gives

$$D^{n+1}(uv) = (D^{n+1}u).v +^{n+1} c_1 (D^n u).Dv +^{n+1} c_2 (D^{n-1}u).(D^2 v) + ...$$
$$..... +^{n+1} c_{r+1} D^{n-r}u.D^{r+1} v + ... + u.D^{n+1} v. \qquad \text{.............(4)}$$

From (4) we see that if the theorem is true for any value of n, it is also true for the next value of n. But we have already seen that the theorem is true for n =1.Hence is must be true for n =2 and so for n =3, and so on. Thus the Leibnitz's theorem is true for all positive integral values of n.

Example. Find the nth differential coefficients of

(i) $\sin ax \cos bx$,

(ii) $\log[(ax + b)(cx + d)]$.

Solution.

(i) Let $y = \sin ax \cos bx = \frac{1}{2}[2\sin ax \cos bx] = \frac{1}{2}[2\sin(a+b)x + \sin(a-b)x]$.

we know that $D^n \sin(ax+b) = a^n \sin\left(ax + b + \frac{1}{2}n\pi\right)$.

$$\therefore y^n = \frac{1}{2}\left[(a+b)^n \sin\left\{(a+b)x + \frac{1}{2}n\pi\right\} + (a-b)^n \sin\left\{(a-b)x + \frac{1}{2}n\pi\right\}\right].$$

(ii) Let $y = \log\left[(ax+b)(cx+d)\right] = \log(ax+b) + \log(cx+d)$.

We know that $D^n \log(ax+b) = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$

$$\therefore y_n = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n} + (-1)^{n-1}(n-1)!c^n(cx+d)^{-n}$$

$$= (-1)^{n-1}(n-1)!\left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right].$$

Example. Find the nth derivatives of $\frac{1}{1-5x+6x^2}$.

Solution.

Let $y = \frac{1}{6x^2 - 5x + 1} = \frac{1}{(2x-1)(3x-1)}$.

$$\therefore \frac{1}{6x^2 - 5x + 1} \equiv \frac{A}{2x-1} + \frac{B}{3x-1} \equiv \frac{A(3x-1) + B(2x-1)}{(2x-1)(3x-1)},$$

Putting $x = \frac{1}{2}, 1 = -\frac{B}{3}$, i.e. $B = -3$; putting $x = \frac{1}{2}, A = 2$.

Hence $y = \frac{2}{2x-1} + \frac{3}{3x-1} = 2(2x-1)^{-1} - 3(3x-1)^{-1}$

Therefor $y_n = \frac{d^n}{dx^n}\left[2(2x-1)^{-1}\right] - \frac{d^x}{dx^n}\left[3(3x-1)^{-1}\right]$

Now we apply the formula,

$D^n(ax+b)^{-1} = (-1)^n(n!)(ax+b)^{-n-1}a^n$.

Hence $y_n = 2.2^n(-1)^n(n!)(2x-1)^{-n-1} - 3.3^n(-1)^n(n!)(3x-1)^{-n-1}$.

or $\quad y_n = (-1)^n(n!)\left[\frac{2^{n+1}}{(2x-1)^{n+1}} + \frac{3^{n+1}}{(3x-1)^{n+1}}\right]$.

Example. If $y = \sin ax + \cos ax$, prove that $y^n = a^n\left[1+(-1)^n \sin 2ax\right]^{1/2}$.

Let $y = \sin ax + \cos ax$, then

$$\therefore\ y_n = a^n \sin\left(ax+\frac{1}{2}n\pi\right)+a^n \cos\left(ax+\frac{1}{2}n\pi\right)$$

$$= a^n\left[\left\{\sin\left(ax+\frac{1}{2}n\pi\right)+\cos\left(ax+\frac{1}{2}n\pi\right)\right\}^2\right]^{1/2}$$

$$= a^n\left[1+2\sin\left(ax+\frac{1}{2}n\pi\right)\cos\left(ax+\frac{1}{2}n\pi\right)\right]^{1/2}$$

$$= a^n[1+\sin(2ax+n\pi)]^{1/2} = [1+\sin n\pi\cos 2ax+\cos n\pi \sin 2ax]^{1/2}$$

$$y_n = a^n[1+(-1)^n \sin 2ax]^{1/2} \qquad [\because \sin n\pi = 0 \text{ and } \cos n\pi = (-1)^n]$$

Example. If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, prove that $p+\left(\frac{d^2p}{d\theta^2}\right)=\frac{a^2b^2}{p^3}$.

Solution.

Given $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$ ...(1)

Differentiating both sides of (1) w.r.t θ, we get

$$\therefore\ 2p\left(\frac{dp}{d\theta}\right) = 2(b^2-a^2)\cos\theta\sin\theta \qquad ...(2)$$

Again differentiating both sides of (2) w.r.t θ, we get

$$p\left(\frac{d^2p}{d\theta^2}\right)+\left(\frac{dp}{d\theta}\right)^2 = (b^2-a^2)(\cos^2\theta-\sin^2\theta) \qquad ...(3)$$

Multiplying (3) by p^2 and substituting the value of $\frac{dp}{d\theta}$ form (1) and (3), we get

$$p^3\left(\frac{d^2p}{d\theta^2}\right)+(b^2-a^2)^2\cos^2\theta\sin^2\theta = p^2(b^2-a^2)(\cos^2\theta-\sin^2\theta)$$

or $$p^3\left(\frac{d^2p}{d\theta^2}\right) = (a^2\cos^2\theta+b^2\theta\sin^2\theta)(b^2-a^2)(\cos^2\theta-\sin^2\theta)-(b^2-a^2)^2\cos^2\theta\sin^2\theta$$

or $p^4 + p^3\left(\frac{d^2 p}{d\theta^2}\right) = (b^2 - a^2)[(\cos^2\theta - \sin^2\theta)(a^2\cos^2\theta + b^2\sin^2\theta) - (b^2 - a^2)\cos^2\theta\sin^2\theta]$

$+ (a^2\cos^2\theta - b^2\sin^2\theta)^2$

$=(b^2 - a^2)[(a^2\cos^4\theta - b^2\sin^4\theta) + (a^2\cos^2\theta + b^2\sin^2\theta)^2$

$=b^2a^2(\cos^2\theta + \sin^2\theta) = a^2b^2$

Hence $p + \left(\frac{d^2 p}{d\theta^2}\right) = \frac{a^2b^2}{p^3}$.

Example. Find y_n if $y = x^{n-1}\log x$.

Solution.

By Leibnitz's theoram, we get

$$y_n = D^n(x^{n-1}\log x) = D^n(x^{n-1})\log x + nD^{n-1}(x^{n-1})D\log x + \frac{n(n-1)}{2!}D^{n-2}(x^{n-1})D^2\log x$$

$$+\frac{n(n-1)(n-2)}{3!}D^{n-3}(x^{n-1})D^3\log x + ... + x^{n-1}D^n Logx.$$

Now $D^n x^m = \frac{m!}{(m-n)!}x^{m-n}; D^n x^{n-1} = 0$

$$D^{n-1}x^m = \frac{m!}{(m-n+1)!}x^{m-n+1}; \quad \therefore D^{n-1}x^{n-1} = (n-1)!$$

$$D^{n-2}x^{n-1} = \frac{(n-1)!}{1!}x; D^{n-3}x^{n-1} = \frac{(n-1)!}{2!}x^2$$

and $D^n\log x = (-1)^{n-1}\frac{(n-1)!}{x^n}$

Hence

$$y_n = \left[\begin{array}{l} n(n-1)!\frac{1}{x} + \frac{n(n-1)}{2!}\frac{(n-1)!}{1!}x\left(-\frac{!}{x^2}\right) + \frac{n(n-1)(n-2)}{3!}\frac{(n-1)!}{2!}x^2\frac{2}{x^3} + ... \\ \qquad .... + x^{n-1}\frac{(-1)^{n-1}(n-1)!}{x^n} \end{array}\right]$$

$$= \frac{(n-1)!}{x}[1 - \{1 - {}^nc_1 - {}^nc_2 - {}^nc_3 + ... + (-1)^{n+1}c_n\}]$$

$$= \frac{(n-1)!}{x}[1 - (1-1)^n = \frac{(n-1)!}{x}$$

Aliter. $y = x^{n-1}\log x \qquad \therefore y_1 = (n-1)x^{n-2}\log x + x^{n-2}$.

$\therefore xy_1 = (n-1)x^{n-1}\log x + x^{n-1} = (n-1)y + x^{n-1}$.

Differentiating both sides $(n-1)$times, we have

$$D^{n-2}(xy_1) = (n-1)D^{n-1}y + D^{n-1}x^{n-1}.$$

$$\therefore xy_n + (n-1)y_{n-1} = (n-1)y_{n+1} + (n-1)! \text{ or } y_n = \frac{(n-1)!}{x}$$

Example.

If $\mathrm{y} = a\cos(\log x) + b\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$ and $x^2y_{n+2} + (2n-1)xy_{n+1} + (n^2+1)y_n = 0$.

Solution.

Let $y = a\cos(\log x) + b\sin(\log x)$,

$$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x)\frac{1}{x} \text{ or } xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Now again differentiating both sides, we get

$$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

or $x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or $x^2y_2 + xy_1 = -y$

or $x^2y_2 + xy_1 + y = 0$.

Again differentiating both sides in times by Leibnitz's theorem,

$$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0.$$

or $x^2D^ny_2 + nDx^2D^{n-1}y_2 + \frac{n(n-1)}{2}D^2x^2D^{n-2}y_2 + xD^ny_1 + nD^{n+1}y_1 + y_n = 0$

or $x^2y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$

or $x^2y_{n+2} + (2n-1)xy_{n+1} + (n^2+1)y_n = 0$.

Example If $y = \sin(m\sin^{-1}x)$. prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and deduce that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$.

Solution: Let $y = \sin(m\sin^{-1}x)$.

$y_1 = \cos(m\sin^{-1}x).\dfrac{m}{\sqrt{(1-x^2)}}$. or $(1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$.

or $(1-x^2)y_1^2 = m^2 - m^2\sin^2(m\sin^{-1}x) = m^2 - m^2y^2$

$\therefore\ (1-x^2)y_1^2 + m^2y = m^2$.

Again differentiating both sides, we have

$2y_1y_2(1-x^2) - 2xy_1^2 + 2m^2yy_1 = 0.$ or $y_2(1-x^2)xy_1 + m^2y = 0$.

Now differntiating n time by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2) - xy_{n+1} - ny_n + m^2y_n = 0,$$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$.

To find The nth Derivative When x = 0

Example: Find $(y_n)_0$. if $y = \sin(a\sin^{-1}x)$.

Solution:

Let $y = \sin(a\sin^{-1}x)$. ……..(1)

$\therefore\ y_1 = \cos(a\sin^{-1}x).\dfrac{a}{\sqrt{(1-x^2)}}$,

or $y_1^2(1-x^2) = a^2\cos^2(a\sin^{-1}x) = a^2 - a^2\sin^2(a\sin^{-1}x) = a^2 - a^2y^2$

or $y_1^2(1-x^2) + a^2y^2 - a^2 = 0$. ....(2)

Differentiating (2), we have

$2y_1y_2(1-x^2) - y_1{}^2(-2x) + 2a^2yy_1 = 0$.

or $y_2(1-x^2) + xy_1 + a^2y_1 = 0$ ....(3)

Differentiating (3) n times, we have

$$y_{n+2}(1-x^2) - ny_{n+1}2x - \frac{n(n-1)}{2!}y_n.2 - xy_{n+1} - ny_n + a^2y_n = 0.$$

or $y_{n+2}(1-x^2) - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0$. ....(4)

Putting x = 0 in (1), we get $(y)_0 = 0$.

Putting x = 0 in (2), we get $(y_1)_0 = 0$

Putting x = 0 in (3), we get $(y_2)_0 = 0$ and

Putting x = 0 in (4), we get $(y_{n+2})_0 = (n^2 - a^2)(y_n)_0$

Now putting n = 2 in (5), $(y_6)_0 = (2^2 - a^2)(y_2)_0 = 0.$

Putting n = 4 in (5), $(y_6)_0 = (4^2 - a^2)(y_4)_0 = 0.$

Similarly $(y_8)_0 = 0.$

Thus the derivatives for which n is even are zero

Again, putting $n = 1, (y_3)_0 = (1^2 - a^2).(y_1)_0 (1^2 - a^2)a.$

Now when n is odd. $(y_{n+2})_0 = (n^2 - a^2)(y_n)_0.$

Putting n is place of (n-2) we obtain

$$(y_n)_0 = \left[(n-2)^2 - a^2\right](y_{n-2})_0$$
$$= \left[(n-2)^2 - a^2\right]\left[(n-4)^2 - a^2\right]\left[(n-6)^2 - a^2\right]\text{.........}\left[3^2 - a^2\right]\left[y_3\right]_0$$
$$= \left[(n-2)^2 - a^2\right]\left[(n-4)^2 - a^2\right]\text{.........}\left[3^2 - a^2\right]\left[1^2 - a^2\right].a.$$

Example.

If $y = \tan^{-1} x,$ prove that $(1 + x^2)y_2 + 2xy_1 = 0$ and deduce that $(1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ Hence determine $(y_n)_0$

Solution.

Let $y = \tan^{-1} x$ ....(1)

$$\therefore y_1 = \frac{1}{(1 + x^2)}, \qquad \text{...(2)}$$

or $(1 + x^2)y_1 - 1 = 0.$ ...(3)

Differentiating (3), we get $(1 + x^2)y_2 + 2xy_1 = 0$ ....(4)

Now , differentiating (4) n times by Leibnitz's theorem, we get

$$y_{n+2}(1 + x^2) + ny_{n+1}(2x) + \frac{n(n-1)}{2!}y_n.2 + 2xy_{n+1} + 2ny_n = 0$$

or $(1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ ....(5)

Putting x = 0, in (1),(2) and (4), we get

$(y)_0 = 0, \ \ (y_1)_0 = 1, \ \ (y_2)_0 = 0.$

Also putting x = 0 in (5) we get

$(y_{n+2})_0 = -\left[(n+1)n\right](y_n)_0.$ ....(6)

Putting n-2 in place of n in the foumula (6),we get

$(y_n)_0 = [(n-1)(n-2)](y_{n-2})_0$

$= [-\{(n-1)(n-2)\}][-\{(n-3)(n-4)\}](y_{n-4})_0$

Since from (6), we have $(y_{n-2})_0 = -\{(n-3)(n-4)\}](y_{n-4})_0$

Case I. When n is even, we have

$(y_n)_0 = [-\{[(n-1)(n-2)\}][-\{(n-3)(n-4)\}]....[-(3)(2)](y_2)_0$

$= 0,$ Since $(y_2)_0 = 0.$

Case II. When n is odd, we have

$(y_n)_0 = [-\{[(n-1)(n-2)][-\{(n-3)(n-4)\}]....[-(4)(3)][-(2)(1)(y_1)_0$

$= (-1)^{(n-1)2}(n-1)!,$ since $(y_1)_0 = 1.$

ADDITIONAL PROBLEMS:

1. Find the nth differential coefficient of
   (*i*) $\sin^3 x$
   (*ii*) $\sin x \cos 3x$
   (*iii*) $e^{ax} \cos^2 x \sin x$
   (*iv*) $\dfrac{x^2}{(x+2)(2x+3)}$

2. If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + \left(a^2 + b^2\right) y = 0$

3. If $y = \cos\left(m \sin^{-1} x\right)$, prove that $\left(1 - x^2\right) y_2 - xy_1 + m^2 y = 0$ and $\left(1 - x^2\right) y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

4. If $y = \left(\sin^{-1} x\right)^2$, prove that $\left(1 - x^2\right) y_2 - xy_1 - 2 = 0$ and deduce that $\left(1 - x^2\right) y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

5. If $y = e^{\tan^{-1} x}$, prove that $\left(1 + x^2\right) y_{n+2} + \{2(n+1)x - 1\} y_{n+1} + n(n+1) y_n = 0$