$\oint$ 3 Isomorphic Binary Structures

2-8,16-18,26,27

2. $< \mathcal{Z}, + >$ with $< \mathcal{Z}, + >$ where $\phi(n) = -n$ for $n \in \mathcal{Z}$. **YES**
   $\phi(a+b) = -a - b$ and $\phi(a) + \phi(b) = -a - b$

3. $< \mathcal{Z}, + >$ with $< \mathcal{Z}, + >$ where $\phi(n) = 2n$ for $n \in \mathcal{Z}$. **NO**
   ϕ is not onto $\mathcal{Z}$. There is no $n \in \mathcal{Z} \ni \phi(n) = 1 \in Z'$.

4. $< \mathcal{Z}, + >$ with $< \mathcal{Z}, + >$ where $\phi(n) = n + 1$ for $n \in \mathcal{Z}$. **NO**
   $\phi(a+b) = a + b + 1$ but $\phi(a) + \phi(b) = a + b + 2$

5. $< \mathcal{Q}, + >$ with $< \mathcal{Q}, + >$ where $\phi(x) = \frac{x}{2}$ for $x \in \mathcal{Q}$. **YES**
   $\phi(a+b) = \frac{a+b}{2}$ and $\phi(a) + \phi(b) = \frac{a}{2} + \frac{b}{2}$

6. $< \mathcal{Q}, \cdot >$ with $< \mathcal{Q}, \cdot >$ where $\phi(x) = x^2$ for $x \in \mathcal{Q}$. **NO**
   Not $1-1$: $\phi(a) = \phi(-a)$ but $a \neq -a$

7. $< \mathcal{R}, \cdot >$ with $< \mathcal{R}, \cdot >$ where $\phi(x) = x^3$ for $x \in \mathcal{R}$. **YES**
   $\phi(ab) = (ab)^3$ and $\phi(a) \cdot \phi(b) = a^3 \cdot b^3 = (ab)^3$

8. $< M_2(\mathcal{R}), \cdot >$ with $< \mathcal{R}, \cdot >$ where $\phi(A)$ is the determinant of the matrix A. **NO**
   Not $1-1$.
   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad |A| = |B| \text{ but } A \neq B.$$

16. The map $\phi : \mathcal{Z} \to \mathcal{Z}$ defined by $\phi(n) = n + 1$ for $n \in \mathcal{Z}$ is one to one and onto $\mathcal{Z}$. Give the definition of a binary operation $*$ on $\mathcal{Z}$ such that ϕ is an isomorphism mapping.

    a. $< \mathcal{Z}, + >$ onto $< \mathcal{Z}, * >$
       For ϕ to be an isomorphism, we must have
       $m * n = \phi(m-1) * \phi(n-1) = \phi((m-1) + (n-1)) = \phi(m + n - 2) = m + n - 1.$
       The identity element $\phi(0) = 1$.

    b. $< \mathcal{Z}, * >$ onto $< \mathcal{Z}, + >$
       Using the fact that ϕ^{-1} is an isomorphism, we must have
       $m * n = \phi^{-1}(m+1) * \phi^{-1}(n+1) = \phi^{-1}((m+1) + (n+1)) = \phi^{-1}(m+n+2) = m+n+1.$
       The identity element is $\phi^{-1}(0) = -1$.

17. The map $\phi : \mathcal{Z} \to \mathcal{Z}$ defined by $\phi(n) = n + 1$ for $n \in \mathcal{Z}$ is one to one and onto $\mathcal{Z}$. Give the definition of a binary operation $*$ on $\mathcal{Z}$ such that ϕ is an isomorphism mapping.

    a. $< \mathcal{Z}, \cdot >$ onto $< \mathcal{Z}, * >$
       For ϕ to be an isomorphism, we must have
       $m * n = \phi(m-1) * \phi(n-1) = \phi((m-1) \cdot (n-1)) = \phi(mn - m - n + 1) = mn - m - n + 2.$
       The identity element is $\phi(1) = 2$.

    b. $< \mathcal{Z}, * >$ onto $< \mathcal{Z}, \cdot >$
       Using the fact that ϕ^{-1} must also be an isomorphism, we must have
       $m * n = \phi^{-1}(m+1) * \phi^{-1}(n+1) = \phi^{-1}((m+1) \cdot (n+1)) = \phi^{-1}(mn+m+n+1) = mn+m+n.$
       The identity element is $\phi^{-1}(1) = 0$.

18. The map $\phi : \mathcal{Q} \to \mathcal{Q}$ defined by $\phi(x) = 3x - 1$ for $x \in \mathcal{Q}$ is one to one and onto $\mathcal{Q}$. Give the definition of a binary operation $*$ on $\mathcal{Q}$ such that ϕ is an isomorphism mapping.

   a. $< \mathcal{Q}, + >$ onto $< \mathcal{Q}, * >$
   For ϕ to be an isomorphism, we must have
   $a * b = \phi\left(\frac{a+1}{3}\right) * \phi\left(\frac{b+1}{3}\right) = \phi\left(\frac{a+1}{3} + \frac{b+1}{3}\right) = \phi\left(\frac{a+b+2}{3}\right) = a + b + 1.$
   The identity element is $\phi(0) = -1$.

   b. $< \mathcal{Q}, * >$ onto $< \mathcal{Q}, + >$
   Using the fact that ϕ^{-1} must also be an isomorphism, we must have
   $a * b = \phi^{-1}(3a-1) * \phi^{-1}(3b-1) = \phi^{-1}((3a-1)+(3b-1)) = \phi^{-1}(3a+3b-2) = a+b-\frac{1}{3}.$
   The identity element is $\phi^{-1}(0) = \frac{1}{3}$.

26. Since f is a bijection, f^{-1} is a bijection also. It needs only to be shown that f^{-1} is a homomorphism.
Since f is a homomorphism, we know for $a, b \in S$, $f(a) = x$ and $f(b) = y$ for $x, y \in S'$. We also know that $f(a * b) = f(a) *' f(b)$.
We want to show that $f^{-1}(x *' y) = f^{-1}(x) * f^{-1}(y)$.
Consider the following:

$$\begin{aligned} f^{-1}(x *' y) &= f^{-1}(f(a) *' f(b)) \\ &= f^{-1}(f(a * b)) \\ &= a * b \\ &= f^{-1}(f(a)) * f^{-1}(f(b)) \\ &= f^{-1}(x) * f^{-1}(y) \end{aligned}$$

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A second way to go:
1-1: Suppose $\phi^{-1}(a') = \phi^{-1}(b')$ for all $a', b' \in S'$. Then $a' = \phi(\phi^{-1}(a')) = \phi(\phi^{-1}(b')) = b'$. So, ϕ^{-1} is 1-1.
Onto: Let $a \in S$. Then $\phi^{-1}(\phi(a)) = a$, so ϕ^{-1} maps S' onto S.
Homomorphism Property: Let $a', b' \in S'$. Now,

$$\phi(\phi^{-1}(a' *' b')) = a' *' b'$$

Because ϕ is an isomorphism,

$$\phi(\phi^{-1}(a') * \phi^{-1}(b')) = \phi(\phi^{-1}(a')) *' \phi(\phi^{-1}(b')) = a' *' b'$$

also. Because ϕ is 1-1, we can conclude that the operation is preserved.

27. Onto: We know there is a $y \in S' \ni x \in S \Leftrightarrow \phi(x) = y$. We also know there is a $z \in S'' \ni y \in S' \Leftrightarrow \psi(y) = z$. So, $\exists z \in S''$ and $x \in S \ni \psi(\phi(x)) = z$.
$1-1$: For $x, y \in S$, $\phi(x) = \phi(y)$ only when $x = y$ and for $\phi(x), \phi(y) \in S'$, $\psi(\phi(x)) = \psi(\phi(y))$ only when $\phi(x) = \phi(y)$ which is only when $x = y$.
Homomorphism: We want to show $\psi(\phi(x * y)) = \psi(\phi(x)) *'' \psi(\phi(y))$.
$\psi(\phi(x * y)) = \psi(\phi(x) *' \phi(y)) = \psi(\phi(x)) *'' \psi(\phi(y))$.

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Another way to go:
1-1: Let $a, b \in S$ and suppose $(\psi \circ \phi)(a) = (\psi \circ \phi)(b)$. Then $(\psi(\phi(a)) = (\psi(\phi(b))$. Because ψ is 1-1, we conclude that $\phi(a) = \phi(b)$. Because ϕ is 1-1, it must be so that $a = b$.
Onto: Let $a'' \in S''$. Because ψ maps S' onto S'', there exists $a' \in S'$ such that $\psi(a') = a''$. Because ψ maps S onto S', there exists $a \in S$ such that $\phi(a) = a'$. Then, $(\psi \circ \phi)(a) = (\psi(\phi(a)) = \psi(a') = a''$, so $\psi \circ \phi$ maps S onto S''.
Homomorphism: Let $a, b \in S$. Since ψ and ϕ are isomorphisms, $(\psi \circ \phi)(a * b) = \psi(\phi(a * b)) = \psi(\phi(a) *' \phi(b)) = \psi(\phi(a)) *'' \psi(\phi(b))$.