l'Hôpital's Rule

L'Hôpital's Rule was stated in the first calculus textbook ever written. It's useful for finding limits for expressions at points where the result is an **indeterminate form**, meaning a mathematical expression which might come out to multiple values.

As an example, the expression 0 × ∞ is an indeterminate form. This can be easily shown using two statements from your grade school geometry classes:

1) Points have no dimension. Specifically, they have no length.
2) A line segment contains an infinite number of points.

So this line segment has an infinite number of points, each of which has no length. However, the line segment has a non-zero and non-infinite length. The same would be true for any line segment, no matter what its length is. In terms of a line segment length, the expression "0 $^{cm}/_{point}$ × ∞ points" could have any real number as an answer.

It is important to note that this is different from an expression which is **undefined**, where there is no real number value that the expression can take. The expression $\frac{2}{0}$ is undefined, not indeterminate, since, if the expression had a value x, we could write:

$$\frac{2}{0} = x$$
$$2 = 0 \cdot x$$

…and there is clearly no real number solution for this equation.

l'Hôpital's Rule is used for indeterminate forms that are quotients, i.e. $\frac{0}{0}$ and $\frac{\infty}{\infty}$:

If for some real number, a, $f(a) = g(a) = 0$, or if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

There are other details of the rule (the limits must exist, etc.) but that's the basic form of it. The rule is true whether the limit of the quotient of the derivatives comes out to a finite real answer or ± infinity. Let's see how this works.

Example 1: Evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$.

Solution: First we must ensure that the expression conforms to l'Hôpital's Rule. If it doesn't, we can't use it. As x approaches 0, both sin x and x approach 0, so this is an indeterminate form. This means we can write:

$$\lim_{x \to 0} \frac{\sin x}{x} \overset{\text{L'H}}{=} \lim_{x \to 0} \frac{\cos x}{1}$$
$$= 1$$

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Notice that is *not* the derivative of $\frac{\sin x}{x}$, which would involve using the Quotient Rule.

The rule may be applied multiple times until a determinate form comes out of the expression:

Example 2:   Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

Solution: This is an expression in the form $\frac{0}{0}$, so we may use l'Hôpital's Rule:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{\sin x}{2x}\qquad\left[\to\frac{0}{0}\right]$$
$$\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{\cos x}{2}$$
$$=\tfrac{1}{2}$$

OTHER INDETERMINATE FORMS: PRODUCTS, DIFFERENCES AND EXPONENTS

While l'Hôpital's Rule is set up specifically for $\frac{0}{0}$ and $\frac{\infty}{\infty}$, it can be adapted for other forms ($0\times\infty$; $\infty-\infty$; 0^0; ∞^0; 1^∞) by rewriting the expression in the limit as a quotient.

If the expression is a **product** ($0\times\infty$**)**, we can invert and divide:

Example 3:   Evaluate $\lim\limits_{x\to 0^+} x\ln x\quad[\to 0\times\infty]$.

Solution:
$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}$$
$$\overset{\text{L'H}}{=}\lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$
$$=\lim_{x\to 0^+}-\frac{x^2}{x}=\lim_{x\to 0^+}(-x)=0$$

If the expression is a **difference** ($\infty-\infty$)**,** we can either find a common denominator or a common factor:

Example 4:   Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\sec x-\tan x\quad[\to\infty-\infty]$

Solution: Find a common denominator:
$$\lim_{x\to\frac{\pi}{2}}\sec x-\tan x=\lim_{x\to\frac{\pi}{2}}\frac{1}{\cos x}-\frac{\sin x}{\cos x}$$
$$=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}\qquad\left[\to\frac{\infty}{\infty}\right]$$
$$\overset{\text{L'H}}{=}\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=\lim_{x\to\frac{\pi}{2}}\cot x=0$$

Example 5: Evaluate $\lim\limits_{x\to\infty} x - \sqrt[3]{x^3 - 1}$ $[\to \infty - \infty]$

Solution: Find a common factor:

$$\lim_{x\to\infty} x - \sqrt[3]{x^3 - 1} = \lim_{x\to\infty} x - \sqrt[3]{x^3 - \frac{x^3}{x^3}}$$

$$= \lim_{x\to\infty} x - \sqrt[3]{x^3\left(1 - \frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} x\left[1 - \sqrt[3]{1 - \frac{1}{x^3}}\right]$$

$$= \lim_{x\to\infty} \frac{1 - \sqrt[3]{1 - \frac{1}{x^3}}}{1/x} \quad \left[\to \frac{0}{0}\right]$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{-\frac{1}{3}\left(1 - \frac{1}{x^3}\right)^{-2/3}\cdot(3)\left(\frac{1}{x^4}\right)}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^4\left(1 - \frac{1}{x^3}\right)^{2/3}} = \lim_{x\to\infty} \frac{1}{x^2\left(1 - \frac{1}{x^3}\right)^{2/3}} = \frac{1}{\infty} = 0$$

For an **exponential expression**, we can use logarithms to rewrite the expression as a product, and then quotient. There are three indeterminate forms you might see: 1^∞, ∞^0 or 0^0. Here is an example:

Example 6: Evaluate $\lim\limits_{x\to 0^+} x^x$.

Solution: We will use the fact that the function e^x is continuous everywhere to put the limit symbol into an exponent. For ease of writing, we'll also be using the other symbol for e^x, which is exp x:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x}$$

e^x is continuous, so...
$$= \exp\left(\lim_{x\to 0^+} \ln x^x\right) = \exp\left(\lim_{x\to 0^+} x \ln x\right)$$

convert the product into a quotient
$$= \exp\left(\lim_{x\to 0^+} \frac{\ln x}{1/x}\right)$$

apply l'Hôpital's Rule and simplify
$$\overset{\text{L'H}}{=} \exp\left(\lim_{x\to 0^+} \frac{1/x}{-1/x^2}\right) = \exp\left(\lim_{x\to 0^+} -\frac{x^2}{x}\right)$$

$$= e^0 = 1$$

EXERCISES

A. Use l'Hôpital's Rule to evaluate these limits:

1) $\lim\limits_{x\to\infty} \dfrac{\ln 2x}{x+3}$

2) $\lim\limits_{x\to 1} \dfrac{x^2-x}{\sin \pi x}$

3) $\lim\limits_{x\to 0} \dfrac{e^x - \cos x}{\tan x}$

4) $\lim\limits_{x\to 0} \dfrac{-e^x+1}{\sin x + x}$

5) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x^3 - x^2}$

6) $\lim\limits_{x\to 1} \dfrac{e^{x-1} + \cos \pi x + \sin(1-x)}{\frac{1}{3}x^3 - x + \frac{2}{3}}$

B. Use l'Hôpital's Rule to evaluate these limits by first converting them to quotient form:

1) $\lim\limits_{x\to\infty} e^{-x} x^3$

2) $\lim\limits_{x\to\infty} (x - \sqrt{x^2+7})$

3) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \csc x\right)$

4) $\lim\limits_{x\to 0^+} -\ln x \sin x$

5) $\lim\limits_{x\to\infty} x^{1/x^2}$

6) $\lim\limits_{x\to 0} x^2 \tan\left(x + \frac{\pi}{2}\right)$

7) $\lim\limits_{x\to 1} x^{\csc \pi x}$

8) $\lim\limits_{x\to 0^+} \tan^x x$

SOLUTIONS

A. (1) 0   (2) −1/π   (3) 1   (4) −½   (5) ½   (6) $\frac{1+\pi^2}{2}$

B. (1) 0   (2) 0   (3) 0   (4) 0   (5) 1   (6) 0   (7) $e^{-1/\pi}$ (8) 1