3.2 Limits and Continuity of Functions of Two or More Variables.

3.2.1 Elementary Notions of Limits

We wish to extend the notion of limits studied in Calculus I. Recall that when we write $\lim_{x \to a} f(x) = L$, we mean that f can be made as close as we want to L, by taking x close enough to a but not equal to a. In this process, f has to be defined near a, but not necessarily at a. The information we are trying to derive is the behavior of $f(x)$ as x gets closer to a.

When we extend this notion to functions of two variables (or more), we will see that there are many similarities. We will discuss these similarities. However, there is also a main difference. The domain of functions of two variables is a subset of $\mathbb{R}^2$, in other words it is a set of pairs. A point in $\mathbb{R}^2$ is of the form (x, y). So, the equivalent of $x \to a$ will be $(x, y) \to (a, b)$. For functions of three variables, the equivalent of $x \to a$ will be $(x, y, z) \to (a, b, c)$, and so on. This has a very important consequence, one which makes computing limits for functions of several variables more difficult. While x could only approach a from two directions, from the left or from the right, (x, y) can approach (a, b) from infinitely many directions. In fact, it does not even have to approach (a, b) along a straight path as shown in figure 3.7. With functions of one variable, one way to show a limit existed, was to show that the limit from both directions existed and were equal ($\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$). Equivalently, when the limits from the two directions were not equal, we concluded that the limit did not exist. For functions of several variables, we would have to show that the limit along every possible path exist and are the same. The problem is that there are infinitely many such paths. To show a limit does not exist, it is still enough to find two paths along which the limits are not equal. In view of the number of possible paths, it is not always easy to know which paths to try. We give some suggestions here. You can try the following paths:

1. Horizontal line through (a, b), the equation of such a path is $y = b$.
2. Vertical line through (a, b), the equation of such a path is $x = a$.
3. Any straight line through (a, b) ,the equation of the line with slope m through (a, b) is $y = mx + b - am$.
4. Quadratic paths. For example, a typical quadratic path through $(0, 0)$ is $y = x^2$.

We will show how to compute limits along a path in the next sections.

While it is important to know how to compute limits, it is also important to understand what we are trying to accomplish. Like for functions of one variable, when we compute the limit of a function of several variables at a point, we are simply trying to study the behavior of that function near that point. The questions we are trying to answer are:

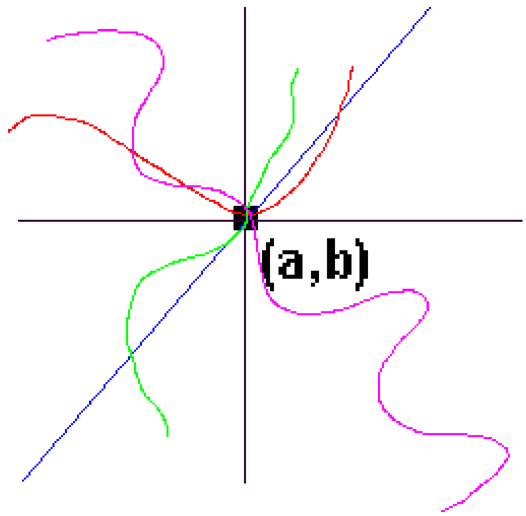


Figure 3.7: Possible paths through (a, b)

1. Does the function behave "nicely" near the point in questions? In other words, does the function seem to be approaching a single value as its input is approaching the point in question?
2. Is the function getting arbitrarily large (going to ∞ or $-\infty$)?
3. Does the function behave erratically, that is it does not seem to be approaching any value?

In the first case, we will say that the limit exists and is equal to the value the function seems to be approaching. In the other cases, we will say that the limit does not exist. We have the following definition:

Definition 3.2.1 *We write* $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ *and we read* ***the limit of*** $f(x, y)$ ***as*** (x, y) ***approaches*** (a, b) ***is*** L*, if we can make* $f(x, y)$ *as close as we want to* L*, simply by taking* (x, y) *close enough to* (a, b) *but not equal to it.*

Remark 3.2.2 *It is important to note that when computing* $\lim\limits_{(x,y)\to(a,b)} f(x, y)$*,* (x, y) *is never equal to* (a, b)*. In fact, the function may not even be defined at* (a, b)*, yet the limit may still exist. While* (a, b) *may not be in the domain of* f*, the points* (x, y) *we consider as* $(x, y) \to (a, b)$ *are always in the domain of* f*.*

Remark 3.2.3 *There are several notation for this limit. They all represent the same thing, we list them below.*

1. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

2. $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = L$

3. *$f(x,y)$ approaches L as (x,y) approaches (a,b).*

We now look at how limits can be computed.

3.2.2 Finding Limits Using the Numerical Method

We try to estimate or "guess" if a limit exists and what its value is by looking at a table of values. Such a table will be more complicated than in the case of functions of one variable. When $(x,y) \to (a,b)$, we have to consider all possible combinations of $x \to a$ and $y \to b$. This usually results in a square table as the ones shown below.

Example 3.2.4 *Consider the function $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$. Use a table of values to "guess" $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.*
We begin by making a table of values of $f(x,y)$ for (x,y) close to $(0,0)$.

$x \diagdown y$	**−1.0**	**−0.5**	**−0.2**	**0**	**0.2**	**0.5**	**1.0**
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1	0.999	0.986	0.829
0	0.841	0.990	1		1	0.990	0.841
0.2	0.829	0.986	0.999	1	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Looking at the table, we can estimate the limit along certain paths. For example, each column of the table gives the function values for a fixed y value. In the column corresponding to $y = 0$, we have the values of $f(x,0)$ for values of x close to 0, from either direction. So we can estimate the limit along the path $y = 0$. In fact, the column corresponding to $y = b$ can be used to estimate the limit along the path $y = b$. Similarly, the row $x = a$ can be used to estimate the limit along the path $x = a$. The diagonal of the table from the top left to the bottom right correspond to values $x = y$. It can be used to estimate the limit along the path $y = x$. The other diagonal, from top right to bottom left corresponds to $y = -x$. So, it can be used to estimate the limit along the path $y = -x$. Looking at the table, it seems that the limit along any of the paths discussed appears to be 1. While this does not prove it for sure, as there are many more paths to consider, this gives us an indication that it might be. We can then try to use other methods we will discuss in the next sections to try to show the limit is indeed 1. It turns out this limit is indeed 1.

Example 3.2.5 *Consider the function* $g(x,y) = \frac{x^2-y^2}{x^2+y^2}$*. Use a table of values to "guess"* $\lim_{(x,y)\to(0,0)} g(x,y)$*.*
We begin by making a table of values of $g(x,y)$ *for* (x,y) *close to* $(0,0)$*.*

$x \diagdown y$	**−1.0**	**−0.5**	**−0.2**	**0**	**0.2**	**0.5**	**1.0**
−1.0	0	0.6	0.923	1	0.923	0.6	0
−0.5	−0.6	0	0.724	1	0.724	0	−0.6
−0.2	−0.923	−0.724	0	1	0	−0.724	−0.923
0	−1	−1	−1		−1	−1	−1
0.2	−0.923	−0.724	0	1	0	−0.724	−0.923
0.5	−0.6	0	0.724	1	0.724	0	−0.6
1	0	0.6	0.923	1	0.923	0.6	0

Looking at the table as indicated in the previous example, we see that the limit along the path $y=0$ *appears to be* 1 *while the limit along the path* $x=0$ *appears to be* -1*. This proves* $\lim_{(x,y)\to(0,0)} g(x,y)$ *does not exist.*

Example 3.2.6 *Consider the function* $h(x,y) = \frac{x^2y}{x^4+y^2}$*. Use a table of values to "guess"* $\lim_{(x,y)\to(0,0)} h(x,y)$*.*
We begin by making a table of values of $h(x,y)$ *for* (x,y) *close to* $(0,0)$*.*

$x \diagdown y$	**−1.0**	**−0.5**	**−0.2**	**0**	**0.2**	**0.5**	**1.0**
−1.0	−0.5	−0.4	−0.1923	0	0.1923	0.4	0.5
−0.5	−0.2352	−0.4	−0.4878	0	0.4878	0.4	0.2352
−0.2	−0.039	−0.079	−0.1923	0	0.1923	0.079	0.039
0	0	0	0		0	0	0
0.2	−0.039	−0.079	−0.1923	0	0.1923	0.079	0.039
0.5	−0.2352	−0.4	−0.4878	0	0.4878	0.4	0.2352
1	−0.5	−0.4	−0.1923	0	0.1923	0.4	0.5

Looking at this table as indicated in the previous examples, it appears that the limit along the paths $x=0$*,* $y=0$*,* $y=x$ *and* $y=-x$ *is* 0*. However, as we will see in the next section, this limit does not exist. In this case, the table would have given the wrong indication.*

In conclusion, we see that tables do not provide as good an answer as in the case of functions of one variable. They can be helpful when the limit does not exist, if the table shows two paths leading to a different limit. However, since the number of paths we can see on the table is limited, they will not, in general tell us for sure if a limit exists. They can still be used to get an idea of whether the limit might exist and what it might be. Given a function, and a limit to compute, if one does not have any idea of what this function does, looking at a table of values might help to point the person in one direction. Usually, solving a problem is easier if one has an idea of what the answer might be. So, while the use of such tables is more limited than in the case of functions of one variable, these tables are not useless.

3.2.3 Finding Limits Using the Analytical Method

Computing limits using the analytical method is computing limits using the limit rules and theorems. We will see that these rules and theorems are similar to those used with functions of one variable. We present them without proof, and illustrate them with examples.

Theorem 3.2.7 (Properties of Limits of Functions of Several Variables) *We list these properties for functions of two variables. Similar properties hold for functions of more variables. Let us assume that L, M, and k are real numbers and that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$, then the following hold:*

1. *First, we have the obvious limits*

$$\begin{aligned} \lim_{(x,y)\to(a,b)} x &= a \\ \lim_{(x,y)\to(a,b)} y &= b \end{aligned}$$

   If c is any constant,

$$\lim_{(x,y)\to(a,b)} c = c$$

2. *Sum and difference rules:*

$$\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$$

3. *Constant multiple rule:*

$$\lim_{(x,y)\to(a,b)} [kf(x,y)] = kL$$

4. *Product rule:*

$$\lim_{(x,y)\to(a,b)} [f(x,y)\, g(x,y)] = LM$$

5. *Quotient rule:*

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M}$$

   provided $M \neq 0$.

6. *Power rule: If r and s are integers with no common factors, and $s \neq 0$ then*

$$\lim_{(x,y)\to(a,b)} [f(x,y)]^{\frac{r}{s}} = L^{\frac{r}{s}}$$

   provided $L^{\frac{r}{s}}$ is a real number. If s is even, we assume $L > 0$.

Theorem 3.2.8 *The above theorem applied to polynomials and rational functions implies the following:*

1. *To find the limit of a polynomial, we simply plug in the point.*

2. *To find the limit of a rational function, we plug in the point as long as the denominator is not* 0.

Example 3.2.9 *Find* $\lim\limits_{(x,y)\to(1,2)} x^6y+2xy$
Combining the rules mentioned above allows us to do the following

$$\begin{aligned} \lim_{(x,y)\to(1,2)} x^6y+2xy &= 1^6 2+2(1)(2) \\ &= 2+4 \\ &= 6 \end{aligned}$$

Example 3.2.10 *Find* $\lim\limits_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2}$
Combining the rules mentioned above allows us to do the following

$$\begin{aligned} \lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2} &= \frac{1^2 1}{1^4+1^2} \\ &= \frac{1}{2} \end{aligned}$$

Remark 3.2.11 *Like for functions of one variable, the rules do not apply when "plugging-in" the point results in an indeterminate form. In that case, we must use techniques similar to the ones used for functions of one variable. Such techniques include factoring, multiplying by the conjugate. We illustrate them with examples.*

Example 3.2.12 *Find* $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$
We cannot plug in the point as we get 0 *in the denominator. We try to rewrite the fraction to see if we can simplify it.*

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y} &= \lim_{(x,y)\to(0,0)} \frac{(x-y)\left(x^2+xy+y^2\right)}{x-y} \\ &= \lim_{(x,y)\to(0,0)} \left(x^2+xy+y^2\right) \\ &= 0 \end{aligned}$$

Example 3.2.13 *Find* $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$
Here, we cannot plug in the point because we get $\dfrac{0}{0}$*, an indeterminate form. Since this is a fraction which involves a radical, we multiply by the conjugate.*

We get:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{\left(x^2-xy\right)\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)} \\
&= \lim_{(x,y)\to(0,0)} \frac{x\left(x-y\right)\left(\sqrt{x}+\sqrt{y}\right)}{x-y} \\
&= \lim_{(x,y)\to(0,0)} \frac{x\left(\sqrt{x}+\sqrt{y}\right)}{1} \\
&= 0
\end{aligned}$$

3.2.4 Limit Along a Path

We have mentioned several times above how important taking the limit along a specific path might be. In particular, one way to prove that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist is to prove that this limit has different values along two different paths. We now look at several examples to see how this might be done. In general, you need to remember that specifying a path amounts to giving some relation between x and y. When computing the limit along this path, use the relation which defines the path. For example, when computing the limit along the path $y=0$, replace y by 0 in the function. If computing the limit along the path $y=x$, replace y by x in the function. And so on...

Make sure that the path you select goes through the point at which we are computing the limit.

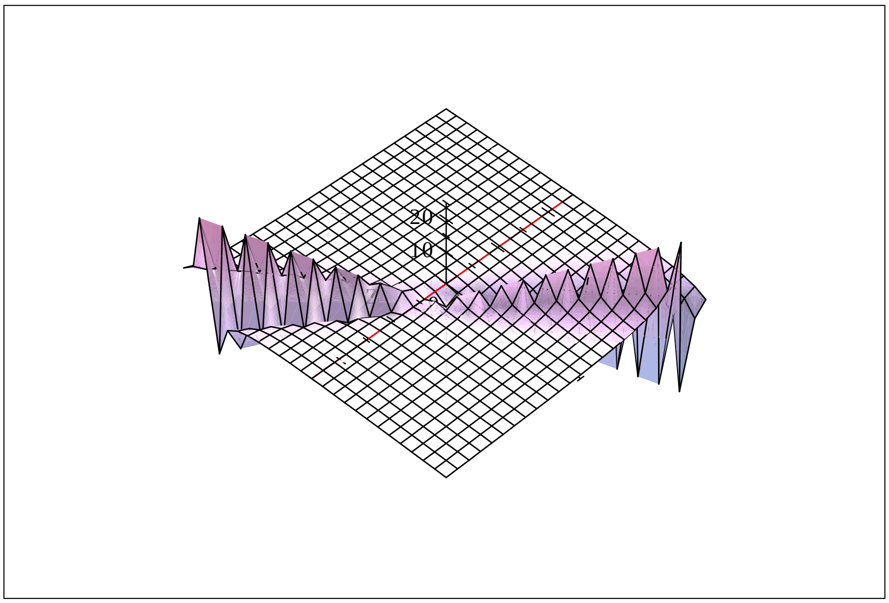
Example 3.2.14 *Consider the function $f(x,y)=\frac{y}{x+y-1}$. The goal is to try to find $\lim\limits_{(x,y)\to(1,0)} \frac{y}{x+y-1}$.*
You may remember from Calculus I that in many cases, to compute a limit we simply plugged-in the point. If you try to do this here, you obtain $\frac{0}{0}$ which is an indeterminate form. It does not mean the limit does not exist. It means that you need to study it further. We will do this by looking at the limit along various paths. As mentioned in the introduction, some obvious paths we might try are the path $x=1$ and $y=0$.

1. *Limit along the path $y=0$. First, we find what the function becomes along this path. We will use the notation $\left.\frac{y}{x+y-1}\right|_{y=0}$ to mean $\frac{y}{x+y-1}$ along the path $y=0$ and $\lim\limits_{\substack{(x,y)\to(1,0)\\ along\ y=0}} \frac{y}{x+y-1}$ to mean $\lim\limits_{(x,y)\to(1,0)} \frac{y}{x+y-1}$ along the path $y=0$. We have:*

$$\begin{aligned}
\left.\frac{y}{x+y-1}\right|_{y=0} &= \frac{0}{x-1} \\
&= 0
\end{aligned}$$

Also, note that along the path $y = 0$, y is constant hence $(x, y) \to (1, 0)$ can be replaced by $x \to 1$. Therefore

$$\begin{aligned} \lim_{\substack{(x,y)\to(1,0) \\ along\ y=0}} \frac{y}{x+y-1} &= \lim_{x\to 1} 0 \\ &= 0 \end{aligned}$$

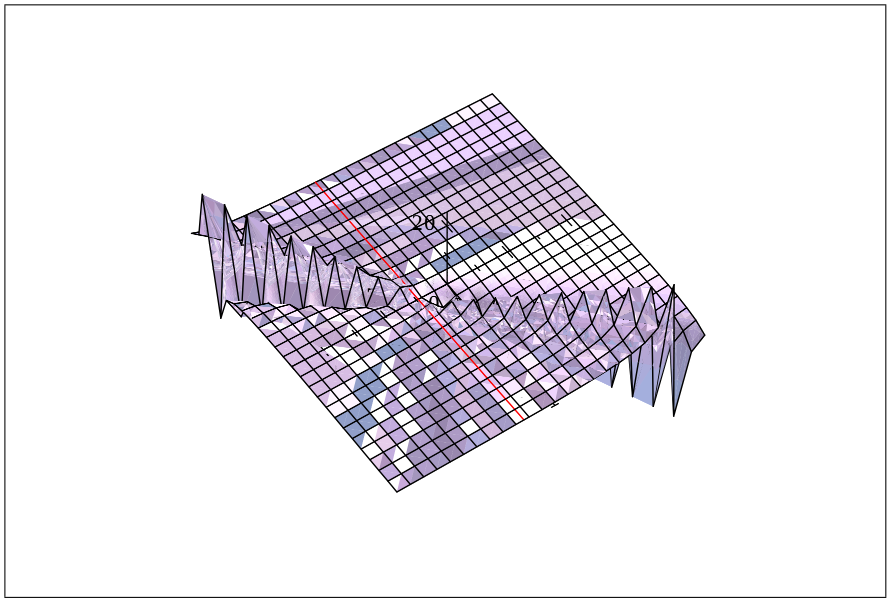


2. *Limit along the path $x = 1$. We have:*

$$\begin{aligned} \left.\frac{y}{x+y-1}\right|_{x=1} &= \frac{y}{1+y-1} \\ &= \frac{y}{y} \\ &= 1 \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\substack{(x,y)\to(1,0) \\ along\ x=1}} \frac{y}{x+y-1} &= \lim_{y\to 0} 1 \\ &= 1 \end{aligned}$$

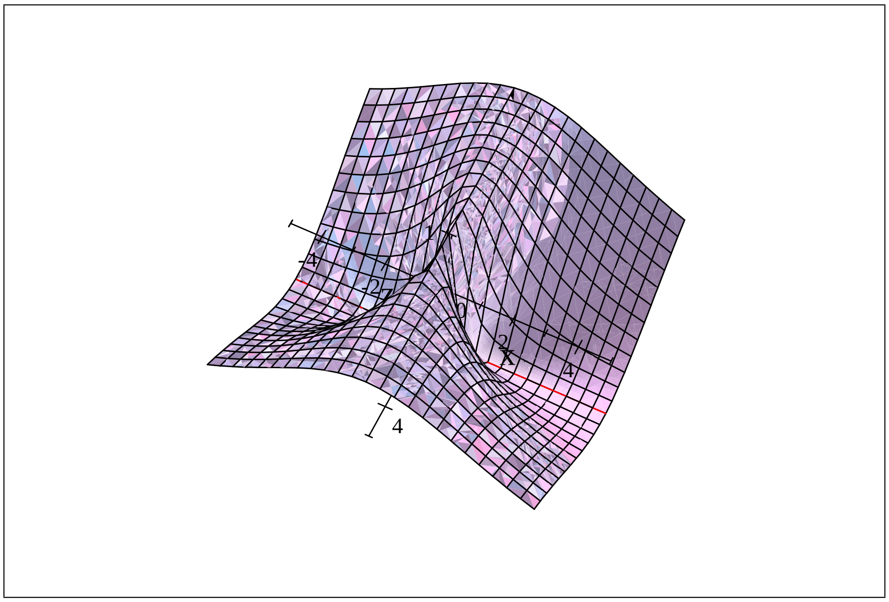


3. *Conclusion: The limits are different, therefore* $\lim\limits_{(x,y)\to(1,0)} \frac{y}{x+y-1}$ *does not exist.*

Example 3.2.15 *Consider the function* $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$*. The goal is to try to find* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$*.*
As mentioned in the introduction, some obvious paths we might try are the path $x = 0$ *and* $y = 0$*. Note that we can also combine both computations (finding what the function is along the path and finding the limit).*

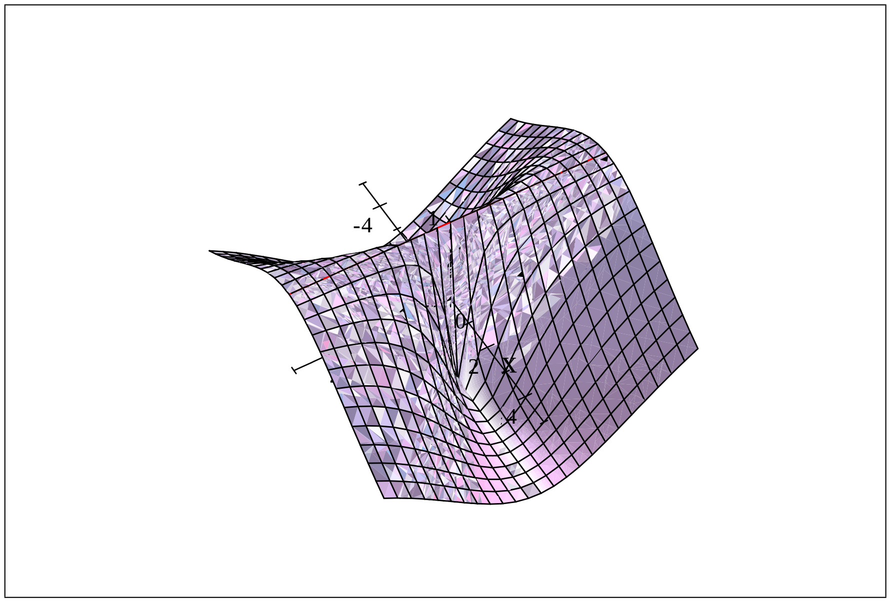
1. *Limit along the path* $x = 0$*. Along this path, we have*

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ x=0}} \frac{x^2-y^2}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \left.\frac{x^2-y^2}{x^2+y^2}\right|_{x=0} \\
&= \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} \\
&= \lim_{(x,y)\to(0,0)} -1 \\
&= -1
\end{aligned}$$



2. *Limit along the path $y = 0$. Along this path, we have*

$$
\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ y=0}} y &= 0\frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left.\frac{x^2-y^2}{x^2+y^2}\right|_{y=0} \\
&= \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} \\
&= \lim_{(x,y)\to(0,0)} 1 \\
&= 1
\end{aligned}
$$



3. *Conclusion: The limits are different, therefore* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ *does not exist.*

Example 3.2.16 *Prove that* $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ *does not exist.*
First, we try the limit along the paths $x = 0$ *and* $y = 0$*. The user will check that both limits are* 0*. Next, we try along the path* $y = x$*. We get*

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ y=x}} \frac{xy}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \left.\frac{xy}{x^2+y^2}\right|_{y=x} \\
&= \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} \\
&= \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} \\
&= \lim_{(x,x)\to(0,0)} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

We obtained a different limit. So, $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ *does not exist.*

Example 3.2.17 *Prove that* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ *does not exist.*
You will recognize this function, it is the function in the third table we did earlier. From the table, it appeared that the limit along the paths $x = 0$*,* $y = 0$*,*

and $y=x$ was 0. Yet, you were told the limit did not exist. We prove it here. The reader will check that if we compute the limit along the paths $x=0$, $y=0$, $y=x$, we obtain 0 every time. In fact, the limit along any straight path through $(0,0)$ is 0. The equation of such a path is $y=mx$. Along this path, we get

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ y=mx}} \frac{x^2y}{x^4+y^2} &= \lim_{(x,y)\to(0,0)} \left.\frac{x^2y}{x^4+y^2}\right|_{y=mx} \\
&= \lim_{(x,mx)\to(0,0)} \frac{mx^3}{x^4+m^2x^2} \\
&= \lim_{x\to 0} \frac{mx^3}{x^2\left(x^2+m^2\right)} \\
&= \lim_{x\to 0} \frac{mx}{x^2+m^2} \\
&= 0
\end{aligned}$$

However, we will get a different answer along the path $y=x^2$.

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ y=x^2}} \frac{x^2y}{x^4+y^2} &= \lim_{(x,y)\to(0,0)} \left.\frac{x^2y}{x^4+y^2}\right|_{y=x^2} \\
&= \lim_{(x,x^2)\to(0,0)} \frac{x^4}{x^4+x^4} \\
&= \lim_{x\to 0} \frac{x^4}{2x^4} \\
&= \lim_{x\to 0} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

This proves that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist.

3.2.5 Additional Techniques to Find Limits: Change of Coordinates and Squeeze Theorem or Sandwich Theorem

Sometimes, changing coordinates may be useful. Consider the example below.

Example 3.2.18 *Using polar coordinates, find $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.*
Recall that the relationship between a point in polar coordinates (r,θ) with $r\geq 0$ and rectangular coordinates (x,y) is

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

From which, we can see that

$$\begin{aligned} x^2+y^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta+\sin^2\theta\right) \\ &= r^2 \end{aligned}$$

and

$$\begin{aligned} x^3+y^3 &= r^3\cos^3\theta + r^3\sin^3\theta \\ &= r^3\left(\cos^3\theta+\sin^3\theta\right) \end{aligned}$$

Also, saying $(x,y)\to(0,0)$ *is equivalent to saying* $r\to 0^+$*. Hence, we have:*

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} &= \lim_{r\to 0^+} \frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r^2} \\ &= \lim_{r\to 0^+} r\left(\cos^3\theta+\sin^3\theta\right) \\ &= 0 \end{aligned}$$

There is also an equivalent of the squeeze theorem. Suppose we are trying to find $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ given $f(x,y)$ and we suspect the limit might be L.

Theorem 3.2.19 *Suppose that* $|f(x,y)-L| \le g(x,y)$ *for every* (x,y) *inside a disk centered at* (a,b)*,except maybe at* (a,b)*. If* $\lim\limits_{(x,y)\to(a,b)} g(x,y)=0$ *then* $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$*.*

The difficulty with this theorem is that we must suspect what the limit is going to be. This is not too much of a problem. If you have tried a table of values and found that along all the paths the table allows you to investigate, the limit is the same, or if you have tried to compute the limit along different paths and have found the same value every time. Then, you might suspect the limit exists and is the common value you have found. It is this value you would try in the squeeze theorem. The second difficulty is finding the function g. This is done using approximation of the initial function f. How it is done depends on f. We illustrate how to do it with a few examples.

Example 3.2.20 *Find* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *for* $f(x,y)=\frac{x^2y}{x^2+y^2}$*.*
The reader will check that computing this limit along various paths such as $x=0$*,* $y=0$*,* $y=x$ *gives* 0*. So, you might start suspecting the limit exists and is* 0*. We now use the squeeze theorem to try to prove it. In other words, we need to find a function* $g(x,y)$ *such that* $|f(x,y)-0| \le g(x,y)$ *and* $\lim\limits_{(x,y)\to(0,0)} g(x,y)=0$*.*

To find g, we proceed as follows:

$$\begin{aligned}
|f(x,y)-0| &= \left|\frac{x^2y}{x^2+y^2}-0\right| \\
&= \left|\frac{x^2y}{x^2+y^2}\right| \\
&= \frac{\left|x^2y\right|}{\left|x^2+y^2\right|} \\
&= \frac{\left|x^2\right||y|}{x^2+y^2} \text{ since } x^2+y^2\geq 0 \\
&= \frac{x^2|y|}{x^2+y^2}
\end{aligned}$$

we can make a fraction bigger by making its denominator smaller. Thus, we have

$$\begin{aligned}
|f(x,y)-0| &= \frac{x^2|y|}{x^2+y^2} \\
&\leq \frac{x^2|y|}{x^2} \\
&= |y|
\end{aligned}$$

If we let $g(x,y)=|y|$, we see that $\lim\limits_{(x,y)\to(0,0)} g(x,y)=0$. Thus, by the squeeze theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$.

Example 3.2.21 *Find $\lim\limits_{(x,y)\to(1,0)} f(x,y)$ for $f(x,y)=\frac{(x-1)^2\ln x}{(x-1)^2+y^2}$.*
The reader will verify that the limit along the paths $x=1$, $y=0$, $y=x-1$ is always 0. So, we suspect the limit we want might be 0. We now use the squeeze theorem to try to prove it. In other words, we need to find a function $g(x,y)$ such that $|f(x,y)-0|\leq g(x,y)$ and $\lim\limits_{(x,y)\to(0,0)} g(x,y)=0$. To find g, we proceed as follows:

$$\begin{aligned}
|f(x,y)-0| &= |f(x,y)| \\
&= \left|\frac{(x-1)^2\ln x}{(x-1)^2+y^2}\right| \\
&= \frac{(x-1)^2|\ln x|}{(x-1)^2+y^2} \\
&\leq |\ln x|
\end{aligned}$$

Since $|\ln x|\to 0$ as $(x,y)\to(1,0)$, it follows by the squeeze theorem that $\lim\limits_{(x,y)\to(1,0)} f(x,y)=0$.

There is another version of the squeeze theorem. As before, we suppose we are trying to find $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ given $f(x,y)$.

Theorem 3.2.22 *If $g(x,y) \leq f(x,y) \leq h(x,y)$ for all $(x,y) \neq (x_0,y_0)$ in a disk centered at (x_0,y_0) and if $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = L$ then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$.*

Here, the difficulty is to find the two functions g and h which satisfy the inequality and have a common limit. We illustrate this with an example.

Example 3.2.23 *Does knowing that $2|xy| - \frac{x^2y^2}{6} \leq 4 - 4\cos\sqrt{|xy|} \leq 2|xy|$ help you with finding $\lim\limits_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|}$?*
If we divide the inequality we have by $|xy|$, then we will have an inequality involving the function for which we want the limit. If the two outer functions in our new inequality have the same limit, then we will be done. Dividing each side of the given inequality by $|xy|$ which is positive (hence preserves the inequality) gives us

$$\frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} \leq \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} \leq \frac{2|xy|}{|xy|}$$

that is

$$\frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} \leq \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} \leq 2$$

We compute $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|}$. We cannot just plug in the point because we get $\frac{0}{0}$. We will eliminate the absolute value by considering cases.

case 1: *$xy > 0$. In this case, $|xy| = xy$ hence*

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} &= \lim_{(x,y)\to(0,0)} \frac{2xy - \frac{x^2y^2}{6}}{xy} \\
&= \lim_{(x,y)\to(0,0)} \frac{xy\left(2 - \frac{xy}{6}\right)}{xy} \\
&= \lim_{(x,y)\to(0,0)} \left(2 - \frac{xy}{6}\right) \\
&= 2
\end{aligned}$$

case 2: $xy < 0$. *In this case,* $|xy| = -xy$ *hence*

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} &= \lim_{(x,y)\to(0,0)} \frac{-2xy - \frac{x^2y^2}{6}}{-xy} \\
&= \lim_{(x,y)\to(0,0)} \frac{-xy\left(2+\frac{xy}{6}\right)}{-xy} \\
&= \lim_{(x,y)\to(0,0)} \left(2+\frac{xy}{6}\right) \\
&= 2
\end{aligned}$$

in conclusion: $\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} = 2$ *and since* $\lim_{(x,y)\to(0,0)} 2 = 2$*, we are exactly in the situation of the squeeze theorem. We conclude that* $\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} = 2$.

3.2.6 Limits with Maple

To compute $\lim_{(x,y)\to(a,b)} f(x,y)$, use

limit($f(x,y), \{x = a, y = b\}$ *);*

3.2.7 Continuity

Like in Calculus I, the definition of continuity is:

Definition 3.2.24 *A function* $f(x,y)$ *is said to be* ***continuous*** *at a point* (a,b) *if the following is true:*

1. (a,b) *is in the domain of* f.
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ *exists.*
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Definition 3.2.25 *If a function* f *is not continuous at a point* (a,b)*, we say that it is* ***discontinuous*** *at* (a,b).

Definition 3.2.26 *We say that a function* f *is continuous on a set* D *if it is continuous at every point in* D.

Thus, if we know that a function is continuous at a point, to find the limit of the function at the point it is enough to plug-in the point. We now review rules and theorem which allow us to determine if a function is continuous at a point, or where a function is continuous.

Theorem 3.2.27 *The following results are true for multivariable functions:*

1. *The sum, difference and product of continuous functions is a continuous function.*

2. *The quotient of two continuous functions is continuous as long as the denominator is not* 0.

3. *Polynomial functions are continuous.*

4. *Rational functions are continuous in their domain.*

5. *If* $f(x,y)$ *is continuous and* $g(x)$ *is defined and continuous on the range of* f*, then* $g(f(x,y))$ *is also continuous.*

We now look at several examples.

Example 3.2.28 *Is* $f(x,y) = x^2y + 3x^3y^4 - x + 2y$ *continuous at* $(0,0)$? *Where is it continuous?*
$f(x,y)$ *is a polynomial function, therefore it is continuous on* $\mathbb{R}^2$. *In particular, it is continuous at* $(0,0)$.

Example 3.2.29 *Where is* $f(x,y) = \frac{2x-y}{x^2+y^2}$ *continuous?*
f *is the quotient of two continuous functions, therefore it is continuous as long as its denominator is not* 0 *that is on* $\mathbb{R}^2 \setminus \{(0,0)\}$.

Example 3.2.30 *Where is* $f(x,y) = \frac{1}{x^2-y}$ *continuous?*
As above, f *is the quotient of two continuous functions. Therefore, it is continuous as long as its denominator is not* 0. *The denominator is* 0 *along the parabola* $y = x^2$. *Therefore,* f *is continuous on* $\left\{(x,y) \in \mathbb{R}^2 \mid y \neq x^2\right\}$.

Example 3.2.31 *Find where* $\tan^{-1}\left(\frac{xy^2}{x+y}\right)$ *is continuous.*
Here, we have the composition of two functions. We know that $\tan^{-1}$ *is continuous on its domain, that is on* $\mathbb{R}$. *Therefore,* $\tan^{-1}\left(\frac{xy^2}{x+y}\right)$ *will be continuous where* $\frac{xy^2}{x+y}$ *is continuous. Since* $\frac{xy^2}{x+y}$ *is the quotient of two polynomial functions, therefore it will be continuous as long as its denominator is not* 0, *that is as long as* $y \neq -x$. *It follows that* $\tan^{-1}\left(\frac{xy^2}{x+y}\right)$ *is continuous on* $\left\{(x,y) \in \mathbb{R}^2 : y \neq -x\right\}$.

Example 3.2.32 *Find where* $\ln\left(x^2+y^2-1\right)$ *is continuous.*
Again, we have the composition of two functions. $\ln$ *is continuous where it is defined, that is on* $\{x \in \mathbb{R} : x > 0\}$. *So,* $\ln\left(x^2+y^2-1\right)$ *will be continuous as long as* x^2+y^2-1 *is continuous and positive.* x^2+y^2-1 *is continuous on* $\mathbb{R}^2$, *but* $x^2+y^2-1 > 0$ *if and only if* $x^2+y^2 > 1$, *that is outside the circle of radius* 1, *centered at the origin. It follows that* $\ln\left(x^2+y^2-1\right)$ *is continuous of the portion of* $\mathbb{R}^2$ *outside the circle of radius* 1, *centered at the origin.*

Example 3.2.33 *Where is* $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{at} \quad (0,0) \end{cases}$ *continuous?*
Away from $(0,0)$*,* f *is a rational function always defined. So, it is continuous. We still need to investigate continuity at* $(0,0)$*. In an earlier example, we found that* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ *did not exist. Therefore,* f *is continuous everywhere except at* $(0,0)$*.*

Example 3.2.34 *Where is* $f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{at} \quad (0,0) \end{cases}$ *continuous?*
Away from $(0,0)$*,* f *is a rational function always defined. So, it is continuous. We still need to investigate continuity at* $(0,0)$*. In an earlier example, we found that* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$*. Therefore,* f *is also continuous at* $(0,0)$*. It follows that* f *is continuous everywhere.*

Remark 3.2.35 *If the function of the example we just did had been defined such that* $f(0,0) = 1$*, then it would not have been continuous at* $(0,0)$ *since the value of the limit at* $(0,0)$ *would not be the same as the value of the function.*

3.2.8 Problems

Make sure you have read, studied and understood what was done above before attempting the problems.

1. Find $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$
2. Find $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$
3. Find $\lim\limits_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x \tan x$
4. Find $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x-y}$
5. Find $\lim\limits_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$
6. Find $\lim\limits_{(x,y)\to(1,0)} \frac{x \sin y}{x^2+1}$
7. Find $\lim\limits_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y}$ by first rewriting the fraction.
8. Find $\lim\limits_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1}$ by first rewriting the fraction.
9. Find $\lim\limits_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ by first rewriting the fraction.

10. Find $\lim\limits_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$ by first rewriting the fraction.

11. Let $P=(x,y,z)$. Find $\lim\limits_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

12. Let $P=(x,y,z)$. Find $\lim\limits_{P\to(3,3,0)} \left(\sin^2 x+\cos^2 y+\sec^2 z\right)$

13. Let $P=(x,y,z)$. Find $\lim\limits_{P\to(\pi,0,3)} ze^{-2y}\cos 2x$

14. At what points in the plane is $f(x,y)$ continuous?

    (a) $f(x,y)=\sin(x+y)$.
    (b) $f(x,y)=\ln\left(x^2+y^2\right)$.

15. At what points in the plane is $g(x,y)$ continuous?

    (a) $g(x,y)=\sin\frac{1}{xy}$
    (b) $g(x,y)=\frac{x+y}{2+\cos x}$.

16. At what points in space is $f(x,y,z)$ continuous?

    (a) $f(x,y,z)=x^2+y^2-2z^2$.
    (b) $f(x,y,z)=\sqrt{x^2+y^2-1}$

17. At what points in space is $g(x,y,z)$ continuous?

    (a) $g(x,y,z)=xy\sin\frac{1}{z}$.
    (b) $g(x,y,z)=\frac{1}{x^2+z^2-1}$.

18. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$ does not exist.

19. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

20. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ does not exist.

21. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ does not exist.

22. Show that $f(x,y)=\frac{2x^2y}{x^4+y^2}$ has limit 0 along every straight line approaching $(0,0)$.

23. Does knowing that
$$1-\frac{x^2y^2}{3}<\frac{\tan^{-1}xy}{xy}<1$$
tell us anything about
$$\lim_{(x,y)\to(0,0)}\frac{\tan^{-1}xy}{xy}$$

24. Does knowing that
$$\left|\sin\frac{1}{x}\right| \leq 1$$
tell us anything about
$$\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$$

25. Consider the function $f(x,y) = \frac{2xy}{x^2+y^2}$. Suppose we want to find $\lim_{(x,y)\to(0,0)} f(x,y)$.

    (a) Along the path $y = mx$, the function becomes $\frac{2m}{1+m^2}$. Substituting $m = \tan\theta$, show how the value of f varies with the line's angle of inclination.So, the value does depend solely on θ.

    (b) Use the formula obtained in part a to show that the limit along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

26. Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2}$.

27. Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$

28. Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \tan^{-1}\frac{|x|+|y|}{x^2+Y^2}$

3.2.9 Answers

1. Find $\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$
$$\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{5}{2}$$

2. Find $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$
$$\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = 2\sqrt{6}$$

3. Find $\lim_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x\tan x$
$$\lim_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x\tan x = 1$$

4. Find $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$
$$\lim_{(x,y)\to(0,\ln 2)} e^{x-y} = \frac{1}{2}$$

5. Find $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = 1$$

6. Find $\lim_{(x,y)\to(1,0)} \frac{x \sin y}{x^2+1}$

$$\lim_{(x,y)\to(1,0)} \frac{x \sin y}{x^2+1} = 0$$

7. Find $\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y}$ by first rewriting the fraction.

$$\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y} = 0$$

8. Find $\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1}$ by first rewriting the fraction.

$$\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = -1$$

9. Find $\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ by first rewriting the fraction.

$$\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = 2$$

10. Find $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$ by first rewriting the fraction.

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} = \frac{1}{4}$$

11. Let $P = (x, y, z)$. Find $\lim_{P\to(1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

$$\lim_{P\to(1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{19}{12}$$

12. Let $P = (x, y, z)$. Find $\lim_{P\to(3,3,0)} \left(\sin^2 x + \cos^2 y + \sec^2 z\right)$

$$\lim_{P\to(3,3,0)} \left(\sin^2 x + \cos^2 y + \sec^2 z\right) = 2$$

13. Let $P = (x, y, z)$. Find $\lim_{P\to(\pi,0,3)} ze^{-2y} \cos 2x$

$$\lim_{P\to(\pi,0,3)} ze^{-2y} \cos 2x = 3$$

14. At what points in the plane is $f(x,y)$ continuous?

    (a) $f(x,y) = \sin(x+y)$.
    $f(x,y)$ is continuous on $\mathbb{R}^2$.

    (b) $f(x,y) = \ln\left(x^2+y^2\right)$.
    f is continuous for all (x,y) except $(0,0)$.

15. At what points in the plane is $g(x,y)$ continuous?

    (a) $g(x,y) = \sin\frac{1}{xy}$.
    $g(x,y)$ is continuous for all (x,y) not on the x or y-axes.

    (b) $g(x,y) = \frac{x+y}{2+\cos x}$.
    g is always continuous.

16. At what points in space is $f(x,y,z)$ continuous?

    (a) $f(x,y,z) = x^2+y^2-2z^2$.
    Continuous at all points since it is a polynomial.

    (b) $f(x,y,z) = \sqrt{x^2+y^2-1}$
    f is continuous outside the cylinder of radius 1 along the z-axis.

17. At what points in space is $g(x,y,z)$ continuous?

    (a) $g(x,y,z) = xy\sin\frac{1}{z}$.
    g is continuous as long as $z \neq 0$.

    (b) $g(x,y,z) = \frac{1}{x^2+z^2-1}$.
    g is continuous as long as $x^2+z^2 \neq 1$.

18. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$ does not exist.
    Just follow the instructions.

19. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.
    Just follow the instructions.

20. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ does not exist.
    Just follow the instructions.

21. By considering different paths, show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ does not exist.
    Just follow the instructions.

22. Show that $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has limit 0 along every straight line approaching $(0,0)$.
    Just follow the instructions (recall that a line through $(0,0)$ is of the form $y = mx$).

23. Does knowing that $1-\frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$ tell us anything about $\lim\limits_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}$?

    Yes, $\lim\limits_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1$

24. Does knowing that $\left|\sin \frac{1}{x}\right| \leq 1$ tell us anything about $\lim\limits_{(x,y)\to(0,0)} y \sin \frac{1}{x}$?

    Yes, $\lim\limits_{(x,y)\to(0,0)} y \sin \frac{1}{x} = 0$

25. Consider the function $f(x,y) = \frac{2xy}{x^2+y^2}$. Suppose we want to find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

    (a) Along the path $y = mx$, the function becomes $\frac{2m}{1+m^2}$. Substituting $m = \tan\theta$, show how the value of f varies with the line's angle of inclination.
    We obtain
    $$\frac{2m}{1+m^2} = \sin 2\theta$$
    So, the value does depend solely on θ.

    (b) Use the formula obtained in part a to show that the limit along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.
    This follows from the fact that $-1 \leq \sin 2\theta \leq 1$.

26. Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2}$.
    $$\lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = 0$$

27. Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$

    The limit does not exist.

28. Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \tan^{-1} \frac{|x|+|y|}{x^2+Y^2}$
    $$\lim_{(x,y)\to(0,0)} \tan^{-1} \frac{|x|+|y|}{x^2+Y^2} = \frac{\pi}{2}$$

Bibliography

[1] Joel Hass, Maurice D. Weir, and George B. Thomas, *University calculus: Early transcendentals*, Pearson Addison-Wesley, 2012.

[2] James Stewart, *Calculus*, Cengage Learning, 2011.

[3] Michael Sullivan and Kathleen Miranda, *Calculus: Early transcendentals*, Macmillan Higher Education, 2014.