L2: Holomorphic and analytic functions.

Contents:

- Holomorphic functions.
- Cauchy-Riemann conditions.
- Complex differentials and complex partial derivatives.
- Analytic functions.
- The exponential function.

1 Holomorphic functions

A complex function $f(x,y) = u(x,y) + \mathrm{i}\, v(x,y)$ can be also regarded as a function of $z = x + \mathrm{i}\, y$ and its conjugated $\overline{z}$ because we can write

$$f(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2\,\mathrm{i}}) = u(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2\,\mathrm{i}}) + \mathrm{i}\, v(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2\,\mathrm{i}})$$

Classically a *holomorphic function* f was defined as a function f such that the expression $u(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2\,\mathrm{i}}) + \mathrm{i}\, v(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2\,\mathrm{i}})$ does not contain, after simplification, the letter $\overline{z}$ or what is the same f does not depend upon $\overline{z}$.

It is standard to denote $\mathcal{O}(D)$ the set of holomorphic functions on D.

From calculus we know that a given function f do not depends on a variable, say w if the partial derivative $\frac{\partial f}{\partial w}$ is identically zero.

So a simpler way of saying that a function f does not depend on $\overline{z}$ is as follows:

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{1}$$

The problem is that we have no definition of the partial derivative with respect to $\overline{z}$.

Example 1.1. *Any polynomial $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ gives an holomorphic function. Moreover, a convergent power series $f(z) = \sum_{k=0}^{\infty} c_k z^k$ gives a holomorphic function in his disc of convergence. The function $f(x,y) = x^2 + y^2$ is not holomorphic. Why?.*

To give a meaning of the partial derivatives $\frac{\partial f}{\partial \overline{z}}$ and $\frac{\partial f}{\partial z}$ **we assume** that such partial derivatives **do exists** and try to find their definition as follows:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z} \tag{2}$$

and since $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2\,\mathrm{i}}$ we get

$$\mathrm{d}x = \frac{\mathrm{d}z + \mathrm{d}\overline{z}}{2}$$
$$\mathrm{d}y = \frac{\mathrm{d}z - \mathrm{d}\overline{z}}{2\,\mathrm{i}}$$

so we get

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ &= \frac{\partial f}{\partial x}\left(\frac{\mathrm{d}z + \mathrm{d}\overline{z}}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{\mathrm{d}z - \mathrm{d}\overline{z}}{2\,\mathrm{i}}\right) \\ &= \left(\frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2\,\mathrm{i}}\frac{\partial f}{\partial y}\right)\mathrm{d}z + \left(\frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2\,\mathrm{i}}\frac{\partial f}{\partial y}\right)\mathrm{d}\overline{z}\,.\end{aligned}$$

Then we give the following definition of the complex partial derivatives:

$$\begin{cases}\dfrac{\partial f}{\partial z} := \dfrac{1}{2}\dfrac{\partial f}{\partial x} + \dfrac{1}{2\,\mathrm{i}}\dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial \overline{z}} := \dfrac{1}{2}\dfrac{\partial f}{\partial x} - \dfrac{1}{2\,\mathrm{i}}\dfrac{\partial f}{\partial y}\end{cases}$$

If f is differentiable[1] we get

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\frac{\partial f}{\partial z}.(z - z_0) + \frac{\partial f}{\partial \overline{z}}\overline{(z - z_0)} + o(|z - z_0|)}{z - z_0} \tag{3}$$

Thus, we get that $\frac{\partial f}{\partial \overline{z}} = 0$ if and only if the Newton's quotient has a limit. Namely,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = \frac{\partial f}{\partial z}(z_0) \tag{4}$$

Indeed, since $\frac{\partial f}{\partial \overline{z}} = 0$ we get equation (4) when $z \to z_0$ from equation 3 and viceversa since the limit of the quotient

$$\frac{\overline{z}}{z}$$

[1]For example if the partial derivatives of u, v are continuous functions.

does not exist when $z \to 0$.

1.1 Chain rule

Here is simple consequence of (2). Assume that $f \in \mathcal{O}(D)$ and that $\mathrm{z} : [a,b] \to D$ is a curve, i.e. $\mathrm{z}(t) = x(t) + \mathrm{i}\, y(t)$. Then,

$$\frac{\mathrm{d} f(\mathrm{z}(t))}{\mathrm{d}t} = f'(\mathrm{z}(t)).\mathrm{z}'(t)$$

Cauchy-Riemann's conditions

We can define f to be holomorphic if either the limit (4) do exists or equivalently if $\frac{\partial f}{\partial \bar{z}} = 0$.

Now assume that f is holomorphic. Then $\frac{\partial f}{\partial \bar{z}} = 0$. We can write this condition in terms of the partial derivatives of the functions u, v as follows:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{\partial u + \mathrm{i}\, v}{\partial \bar{z}} \\ &= \frac{\partial u}{\partial \bar{z}} + \frac{\partial\, \mathrm{i}\, v}{\partial \bar{z}} \\ &= \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2\,\mathrm{i}}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial\, \mathrm{i}\, v}{\partial x} - \frac{1}{2\,\mathrm{i}}\frac{\partial\, \mathrm{i}\, v}{\partial y} \\ &= \left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}\right) + \mathrm{i}\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial y}\right)\end{aligned}$$

and we obtain the famous *Cauchy-Riemann's* conditions for a holomorphic function $f = u + \mathrm{i}\, v$

$$\begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases} \tag{5}$$

Here is how Riemann originally found the above equations.

By equation (4) the quotient

$$\frac{\mathrm{d}u + \mathrm{i}\,\mathrm{d}v}{\mathrm{d}x + \mathrm{i}\,\mathrm{d}y}$$

is well defined, i.e. independent of $\mathrm{d}x, \mathrm{d}y$. So computing the differential in the numerator we have

$$\frac{(u_x + \mathrm{i}\, v_x)\,\mathrm{d}x + (v_y - \mathrm{i}\, u_y)\,\mathrm{i}\,\mathrm{d}y}{\mathrm{d}x + \mathrm{i}\,\mathrm{d}y}$$

and this is independent of $\mathrm{d}x, \mathrm{d}y$ **if and only if**

$$(u_x + \mathrm{i}\, v_x) = (v_y - \mathrm{i}\, u_y)$$

which are equations (5).

Theorem 1.2. *If f is holomorphic and $f'(z) = 0$ then $df \equiv 0$ and f is locally constant.*

Proof. Since $f = u + \mathrm{i}\, v$ is holomorphic we have $\frac{\partial f}{\partial \bar{z}} = 0$ and $f'(z) \equiv 0$ is equivalent to $\frac{\partial f}{\partial z} = 0$. Then by equation 2 we get $\mathrm{d}f \equiv 0$. This means that the functions u, v are locally constant. □

Interpretations of the CR conditions and harmonic functions

The differential $\mathrm{d}f$ is related to the Jacobian matrix as follows

$$\mathrm{d}f = J_f \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

Now if f is holomorphic the Cauchy-Riemann's equations are:

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

Which means that the Jacobian matrix is given by multiplication against the complex number $u_x + \mathrm{i}\, v_x$. Namely, J_f is the 2×2 matrix associated to the multiplication by $u_x + \mathrm{i}\, v_x$.

From the geometric interpretation of the multiplication by complex numbers we get that near z_0 if $f'(z_0) \neq 0$ the behavior of f is like a rotation followed by an expansion. In particular a point where $f'(z_0)$ can not be a minimum or a maximum of $|f(z)|$.

Here is another interpretation of Cauchy-Riemann's equations. Let us write the gradient ∇u as a complex number

$$\nabla u = u_x + \mathrm{i}\, u_y$$

Then the gradient $\nabla v = v_x + \mathrm{i}\, v_y$ is obtained from ∇u by a $90°$ counterclockwise. That is to say,

$$\nabla v = \mathrm{i}\, \nabla u$$

In general if a vector field is a gradient then its $90°$ counterclockwise rotation is not a gradient.

Another easy but important observation is the harmonicity of the functions u, v. Namely, if $f = u + \mathrm{i}\, v$ is holomorphic then

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \\ \Delta v = v_{xx} + v_{yy} = 0 \end{cases} \tag{6}$$

Indeed, $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$ and so $u_{xx} + u_{yy}$. But also notice that

$$\frac{\Delta}{4} = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2\,\mathrm{i}}\frac{\partial}{\partial y}\right)\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2\,\mathrm{i}}\frac{\partial}{\partial y}\right) = \frac{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}{4}$$

Analytic functions.

A complex function f is called *analytic* if around each point z_0 of its domain the function f can be computed by a convergent power series. More precisely, for each z_0 there exists $\epsilon > 0$ and a sequence of complex numbers $(a_0, a_1, \cdots)$ such that

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots = \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{7}$$

for $|z - z_0| < \epsilon$.

If f is analytic then f and all its derivatives are holomorphic. The derivatives can be computed as the derivatives of a convergent power series,i.e. by deriving term by term. In particular,

$$f^{(n)}(z_0) = \frac{a_n}{n!}$$

which shows that the expression of f as a power series at z_0 is unique.

If the power series (7) is convergent for all $z \in \mathbb{C}$, i.e. not just for $|z - z_0| < \epsilon$, the function f is called entire function.

An important example of entire function is the exponential e^z defined by the power series:

$$e^z = 1 + z + \frac{z^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \ .$$

Notice that the derivative of e^z is itself.

A simple computation shows the Euler's formula

$$e^{\mathrm{i}\,\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$$

for $\theta \in \mathbb{R}$.

The geometric series

$$g(z) = 1 + z + z^2 + z^3 + \cdots$$

is convergent for $|z| < 1$ and so $g(z)$ is holomorphic. If $|z| < 1$ then

$$(1 - z)g(z) = 1 + z + z^2 + \cdots - z - z^2 - \cdots = 1$$

so

$$g(z) = \frac{1}{1-z}\,.$$

The series

$$G(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$$

is also convergent for $|z| < 1$ and $G'(z) = g(z)$.

Notice that $(1-z)e^{G(z)} = 1$ for all $|z| < 1$. So $G(z)$ can be regarded as the logarithm of $\frac{1}{1-z}$.