Points of inflection

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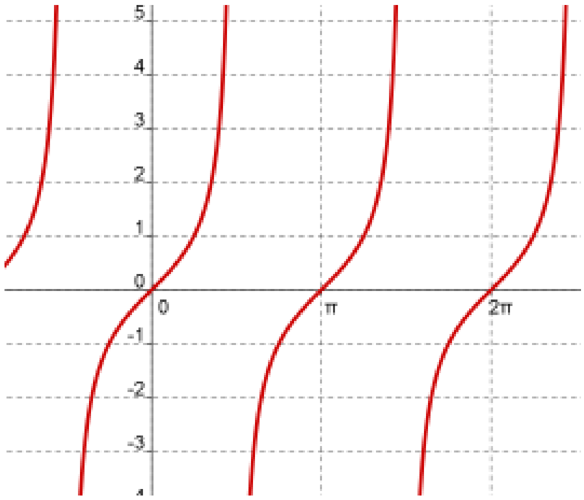
1. Definition

A point of inflection (point of inflexion) $(x_0, f(x_0))$ on a curve is a continuous point at which the function $f(x)$ changes from convex (concave upward) to concave (concave downward) or vice versa as x passes through x_0 .

2. Continuity of the function

If $(x_0, f(x_0))$ is a point of inflection of the function $y = f(x)$, then the function is also continuous at that point. In other words, if (x_0, y_0) is not continuous, it must not be a point of inflection.

Example 1



$y = \tan x$ is concave upward in the interval $\left(n\pi, n\pi + \frac{\pi}{2}\right)$ and concave downward in the interval $\left(n\pi + \frac{\pi}{2}, (n+1)\pi\right)$, where $n \in \mathbb{Z}$.

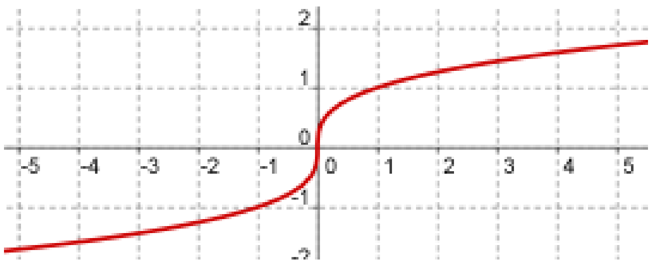
However, at $x = n\pi + \frac{\pi}{2}$, $y = \tan x$ is undefined and therefore these points are not points of inflection.

Readers may check that $(n\pi, 0)$ are points of inflection.

3. First derivative

A point of inflexion of the curve $y = f(x)$ must be continuous point but need not be differentiable there.

Example 2 $y = f(x) = x^{1/3}$



$$f'(x) = \frac{1}{3x^{2/3}}, \quad f''(x) = -\frac{2}{9x^{5/3}}$$

For $x < 0$, $f''(x) > 0 \Rightarrow$ concave upward

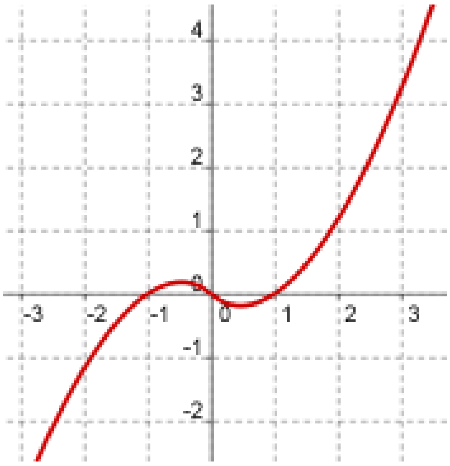
For $x > 0$, $f''(x) < 0 \Rightarrow$ concave downward

Although $f'(0)$ and $f''(0)$ are undefined, (0, 0) is still a point of inflection.

In fact, $y = f(x) = x^{1/3}$ is the inverse function of $y = x^3$. The latter function obviously has also a point of inflection at (0, 0) .

4. Second derivative

Even the first derivative exists in certain points of inflection, the second derivative may not exist at these points.



Example 3 $y = f(x) = x^{5/3} - x$,

$$f'(x) = \frac{5}{3}x^{2/3} - 1, \qquad f''(x) = \frac{10}{9x^{1/3}}$$

If $x < 0$, $f''(x) < 0$ $\Rightarrow$ concave downward.

If $x > 0$, $f''(x) > 0$ $\Rightarrow$ concave upward.

$f'(0) = -1$ exists but $f''(0)$ does not exist.

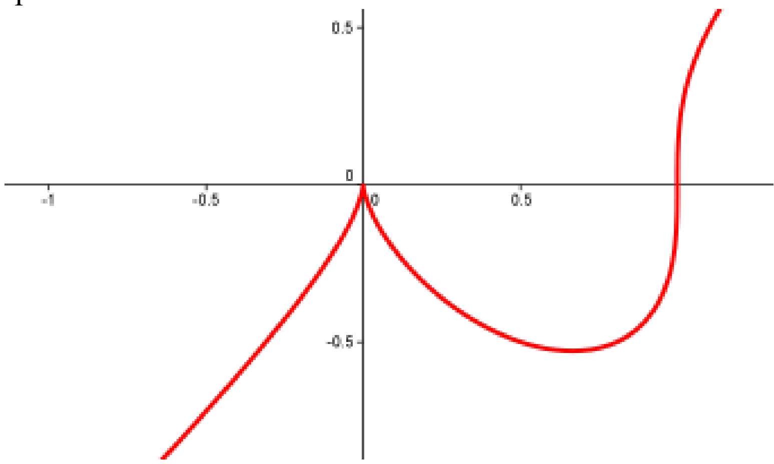
However, (0, 0) is a point of inflection.

Setting the second derivative of a function to zero sometimes **cannot** find out all points of inflection.

Example 4 $y = f(x) = x^{2/3}(x-1)^{1/3}$

$$f'(x) = \frac{3x-2}{3x^{1/3}(x-1)^{2/3}} \quad , \quad f''(x) = -\frac{2}{9x^{4/3}(x-1)^{5/3}}$$

By setting $f''(x) = 0$, we get no solution for points of inflections.



By setting the denominator of $f''(x)$ to 0, we get $x = 0$ or 1.

There is no sign change for $f''(x) = 0$ as x goes through $x = 0$.

$\therefore$ (0, 0) is not a point of inflection.

There is a sign change for $f''(x) = 0$ as x goes through $x = 1$.

$\therefore$ (1, 0) is not a point of inflection.

5. Relative extremum

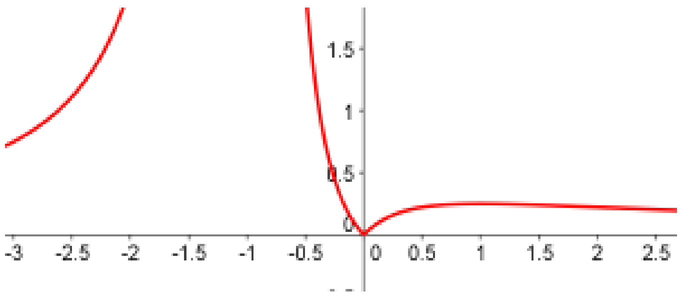
Points of inflection can also be the extremum points (maximum / minimum) at the same time.

Example 4 $y = f(x) = \dfrac{|x|}{(x+1)^2}$

For $x \neq 1$,

$$f(x)=\begin{cases}\dfrac{x}{(x+1)^2} & ,\text{for } x>0\\ -\dfrac{x}{(x+1)^2} & ,\text{for } x<0\end{cases} \qquad f'(x)=\begin{cases}\dfrac{1-x}{(x+1)^2} & ,\text{for } x>0\\ \dfrac{x-1}{(x+1)^2} & ,\text{for } x<0\end{cases} \qquad f''(x)=\begin{cases}\dfrac{2x-4}{(x+1)^2} & ,\text{for } x>0\\ \dfrac{4-2x}{(x+1)^2} & ,\text{for } x<0\end{cases}$$

If $x < 0$, $f''(x) > 0$ $\Rightarrow$ concave upward.
If $x > 0$, $f''(x) < 0$ $\Rightarrow$ concave downward.
$\therefore$ (0, 0) is a point of inflection.



Note that (0, 0) is also a relative minimum point.

The reader may check that:

1. $\left(2, \dfrac{2}{9}\right)$ is another inflection point.

2. $\left(1, \dfrac{1}{4}\right)$ is a relative maximum point.

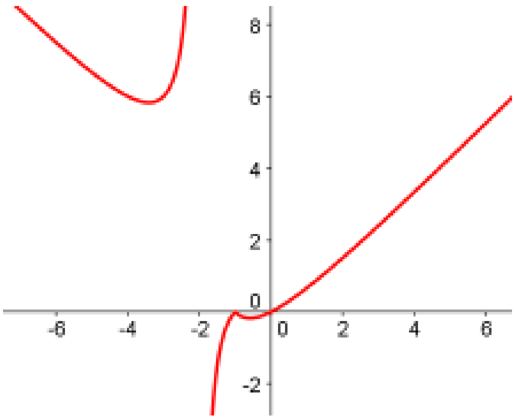
3. Horizontal asymptote : $y = 0$. Vertical asymptote : $x = -1$.

Example 5 $\qquad y = f(x) = \dfrac{x|x+1|}{x+2}$

For $x \neq -2$,

$$f(x)=\begin{cases}\dfrac{x(x+1)}{x+2} & ,\text{for } x>-1\\ -\dfrac{x(x+1)}{x+2} & ,\text{for } x<-1\end{cases} \qquad f'(x)=\begin{cases}\dfrac{x^2+4x+2}{x+2} & ,\text{for } x>-1\\ -\dfrac{x^2+4x+2}{x+2} & ,\text{for } x<-1\end{cases} \qquad f''(x)=\begin{cases}\dfrac{4}{(x+2)^3} & ,\text{for } x>-1\\ -\dfrac{4}{(x+2)^3} & ,\text{for } x<-1\end{cases}$$

If $x < -1$, $f''(x) > 0$ $\Rightarrow$ concave downward.
If $x > -1$, $f''(x) < 0$ $\Rightarrow$ concave upward.
$\therefore$ (-1, 0) is a point of inflection.



Note that (-1, 0) is also a relative minimum point.

The reader may check that:

1. $\left(-2+\sqrt{2}, -3+2\sqrt{2}\right)$ is a relative minimum point .

2. $\left(-2-\sqrt{2}, 3+2\sqrt{2}\right)$ is a relative minimum point.

3. Vertical asymptote is $x = -2$.

4. Oblique asymptotes are $y = x - 1$ (for the positive side) and $y = -x + 1$ (for negative side).

Exercise Find the point(s) of inflection of each of the following curves:

1. $y = f(x) = \sqrt[3]{x^2(x+6)}$ **2.** $x^3 - y^3 = 8$ **3.** $y = f(x) = \sqrt[3]{x^2} + \sqrt[3]{x^2 - 1}$

Derivative help:

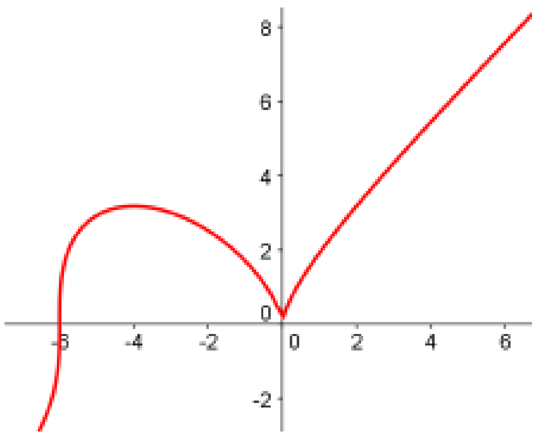
1. $y' = \dfrac{x+4}{x^{1/3}(x+6)^{2/3}}$, $y'' = -\dfrac{8}{x^{4/3}(x+6)^{5/3}}$

2. $y' = \dfrac{x^2}{(x^3-8)^{2/3}}$, $y'' = -\dfrac{16x}{(x^3-8)^{5/3}}$

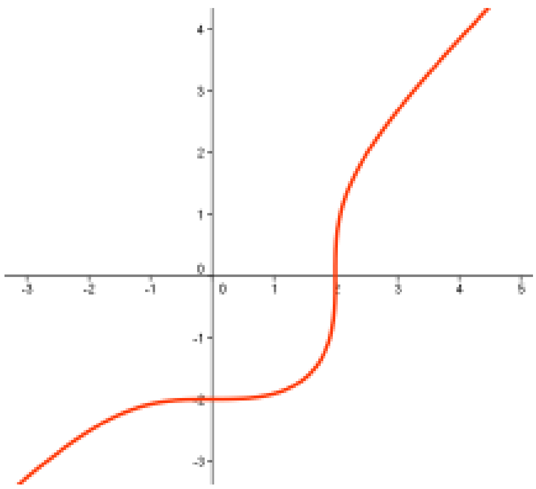
3. $y' = \dfrac{2\left[x^{4/3} + (x^2-1)^{2/3}\right]}{3x^{1/3}(x^2-1)^{2/3}}$, $y'' = -\dfrac{2\left[3x^{4/3} + x^{10/3} - (x^2-1)^{2/3} + x^2(x^2-1)^{2/3}\right]}{9x^{4/3}(x^2-1)^{5/3}}$

Graphical help:

1.



2.



3.