AML710 CAD LECTURE 10

PLANE CURVES

1. Analytical curves
2. Synthetic Curves

PLANE CURVES

The curves are an important part of many engineering disciplines. We recognise curves entirely confined to a plane as plane curves as opposed to the curves existing in 3D spaces.

The curves arise because of the fact that we don't have only polyhedral objects in the real world but there are many curved objects also. To describe such curved objects and their boundaries we need different types of curves.

A curve may result as a solution of an algebraic equation f(x,y)=0 in a plane or as the solution of an equation like g(x,y,z)=0 in space.

ANALYTIC CURVES

Mathematically the curves can be described using algebraic equations or in terms of a parameter (parametric).

The analytical curves can be

- Explicit

y= f(x) => for every x there is one value

Eg: *y=mx+C*

Can we represent a circle (or any closed curve) by such equation?

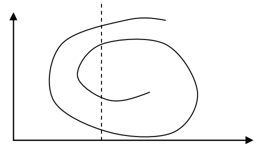
- Implicit (multi valued or closed functions)

f(x,y) = 0 => multi-valued or closed curves

Eg:

$$x^2 + y^2 - r^2 = 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Curve Fitting and Curve Fairing

When we have a given set of points through which a curve has to pass through, essentially we are fitting a curve to the data points. An analytic equation can be written for the curve we have fit through this data. Any point on the curve can be determined by interpolation

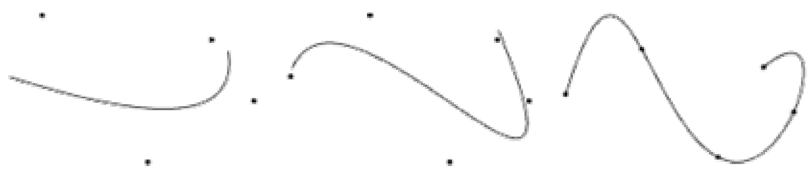
On the other hand, the curve fairing technique is used when an approximate relation of curve is sufficient instead of an accurate one.

We use least square or some other method to minimize the y^2 deviations. Such techniques give an approximate curve

$$y = ax^b; y = ae^{bx}; \; y = c_1 + c_2 x + c_3 x^2 + ... + c_{n+1} x^n$$

Curve Fitting

Here a curve fitting is done using polynomials of order n-3 to n-1 where n is the number of points



The order of the polynomial used directly influences the shape of the curve

ANALYTICAL AND PARAMETRIC REPRESENTATIONS

Analytical Representations

- Precision
- Compact Storage – we store only the equations
- Ease of Interpolation
- Slope and radius of curvature can be determined easily
- Any point on the curve can be precisely determined

Parametric Representations

- The slope of the curve is represented by tangent vectors
- Infinite slope results when one of the components of tangent vector is zero
- As parameter is used, parametric representation is independent of axis
- The curve end points and length are fixed by the range

PARAMETRIZATION

Parametrization is a way to write a function so that all the coordinates (or variables) depend on the same variable.

Example:

If we have a function z = f[x,y], and if the parameter is "t" where the x-coordinate is expressible as g[t], and the y-coordinate is expressible as h[t], we say we can write the function coordinate-wise as {x[t], y[t], z[t]}. We reduce the problem of two variables to that of one input variable (t).

Consider the parametrization of a unit circle x^2+y^2=1 as:

$$P(t)=[x \quad y] \quad where \quad x = x(t) \,\&\, y = y(t)$$

Is there unique way of parametrizing the given function?

PARAMETRIC CURVES

In parametric form each coordinate of a point is represented as a function of a single parameter, say t.

$$P(t)=[x \quad y] \quad where \quad x = x(t) \,\&\, y = y(t)$$

The derivative or tangent vector on the curve is given by

$$P'(t)=[x'(t) \quad y'(t)]$$

The slope of the curve dy/dx –

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{y'(t)}{x'(t)}$$

Infinite slope results when one of the components of tangent vector is zero

As a single parameter is used, parametric representation is independent of axis

The curve end points and length are fixed by the parametric range

Example of parametric curves

A simplest parametric curve, a straight line in single parameter t, is given as

$$P(t) = P_1 + (P_2 - P_1)t \qquad 0 \le t \le 1$$

The components of P(t) in parametric form are:

$$x(t) = x_1 + (x_2 - x_1)t \qquad 0 \le t \le 1$$

$$y(t) = y_1 + (y_2 - y_1)t$$

Ex: Determine the line segment between the position vectors (1 2) (4 3). Also determine the slope and tangent vector.

$$P(t) = P_1 + (P_2 - P_1)t = (1 \quad 2) + [(4 \quad 3) - (1 \quad 2)]t$$

$$= (1 \quad 2) + (3 \quad 1)t$$

$$x(t) = x_1 + (x_2 - x_1)t = 1 + 3t;$$

$$y(t) = y_1 + (y_2 - y_1)t = 2 + t$$

The tangent vector and slope are:

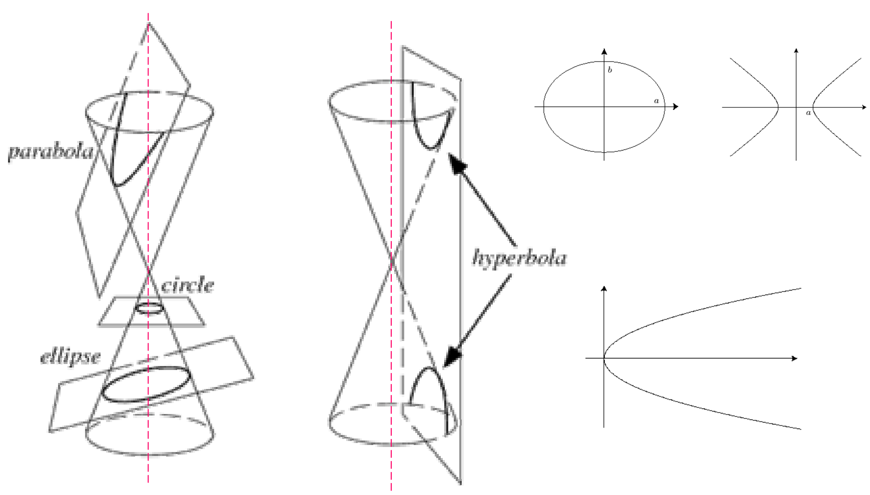
$$P'(t) = [x'(t) \quad y'(t)] = [3 \quad 1] = 3i + j$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = 1/3$$

CONIC SECTIONS

The general second-degree equation for conic sections is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$



CONIC SECTIONS: Projective geometry

The general second-degree equation for conic sections is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

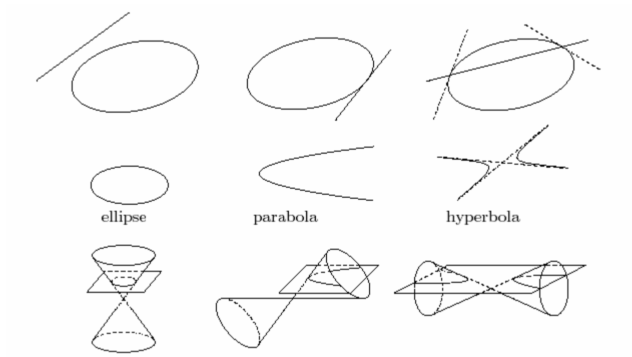


Fig. 19

Taken from *Geometry* by M. Audin

CONIC SECTIONS

The general second-degree equation for conic sections is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

By defining coefficients a, b, c, d, e and f we get variety of conic sections.

If the curve is defined with respect to a local coordinates and passes through origin, then f=0.

Suitable geometric boundary conditions are used to establish curves through specific points

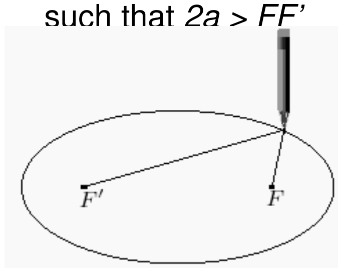
Eg.: c=1, we need 5 independent B.Cs to define the curve between two points. These could be 1. Position of two end points, 2. Slope at these points and 3. Another intermediate point through which the curve must pass.

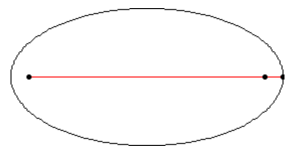
CONIC SECTIONS

Ellipse/circle and hyperbola are central conics and parabola is not. Ellipse and circle form a compact whereas parabola and hyperbola extend to infinity.

How can we draw an ellipse?

Hint: An ellipse with foci *F* and *F'* is locus of a point *M* such that *(MF + MF')=2a* for some non-negative number *a* such that *2a > FF'*



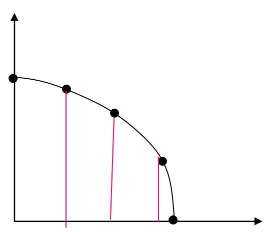


*Source: mathworld.**wolfram**.com*

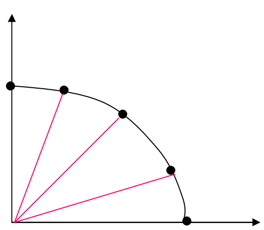
Representation of a circle

A non-parametric representation of a circular arc in the first quadrant is

$$y = \pm\sqrt{1-x^2} \qquad 0 \le x \le 1$$



- Parametric representation



$$P(\theta) = [x \quad y] = [\cos\theta \quad \sin\theta] \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

- Alternatively the following parametric form can be used

$$x = \frac{1-t^2}{1+t^2};\ y = \frac{2t}{1+t^2} \qquad 0 \le t \le 1$$

Algorithmic Representation of a circle

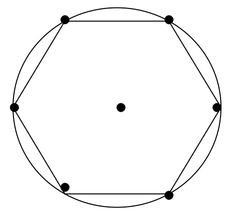
What is the problem with the scheme just discussed?

Computationally intensive because of repeated calculations

What is the solution?

- An improved algorithm due to L.B.Smith (1969)

$$x_{i+1} = r\cos(\theta_i + \delta\theta), \quad y_{i+1} = r\sin(\theta_i + \delta\theta)$$



Where θ_i is the parameter that determines (x_i y_i)

$$x_{i+1} = r(\cos\theta_i \cos\delta\theta - \sin\theta_i \sin\delta\theta);$$

$$y_{i+1} = r(\cos\theta_i \sin\delta\theta + \cos\delta\theta \sin\theta_i)$$

We know that

$$x_i = r\cos\theta_i; \quad y_i = r\sin\theta_i$$

This results in

$$x_{i+1} = x_i \cos\delta\theta - y_i \sin\delta\theta;$$

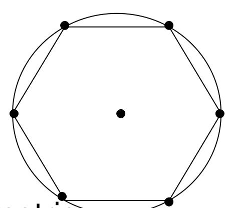
$$y_{i+1} = x_i \sin\delta\theta + y_i \cos\delta\theta \qquad \text{where } \delta\theta = \tfrac{2\pi}{(n-1)}$$

Representation of a circle

Notes:

1. The quantity $\delta\theta$ is constant. Therefore $\cos\delta\theta$ and $\sin\delta\theta$ need to be calculated only once and stored.

2. Only 6 inner loop operations are involved
   - (a) 4 multiplications
   - (b) one subtraction
   - (c) one addition



3. The result is comparable to the simple parametric representation while achieving better efficiency

4. A non-origin centered circle can be generated by translating the origin centered circle.

- Alternatively a unit circle can be used to obtain any circle of desired radius by suitably scaling and translation.

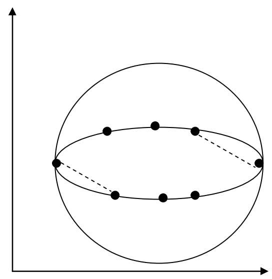
Parametric Representation of an ellipse

Simple relations:

$$x = a\cos\theta; \qquad y = b\sin\theta \qquad 0 \le \theta \le 2\pi \qquad (1)$$

- An improved algorithm due to I.B.Smith (1969)

$$x_{i+1} = a\cos(\theta_i + \delta\theta), \quad y_{i+1} = b\sin(\theta_i + \delta\theta)$$



Where θ_i is the parameter that determines (x_i y_i)

$$x_{i+1} = a(\cos\theta_i \cos\delta\theta - \sin\theta_i \sin\delta\theta);$$

$$y_{i+1} = b(\cos\theta_i \sin\delta\theta + \cos\delta\theta \sin\theta_i)$$

Substitute from (1) $x_i = a\cos\theta_i; \quad y_i = b\sin\theta_i$

This results in

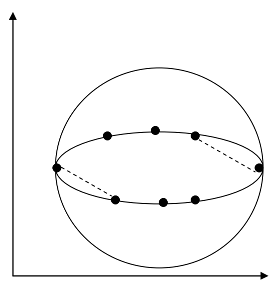
$$x_{i+1} = x_i \cos\delta\theta - \tfrac{a}{b} y_i \sin\delta\theta);$$

$$y_{i+1} = \tfrac{b}{a} x_i \sin\delta\theta + y_i \cos\delta\theta) \quad \text{where } \delta\theta = \tfrac{2\pi}{(n-1)}$$

Representation of an ellipse

Notes:

1. If a=b it reduces to the case of a circle
2. quantity $\delta\theta$ is constant. Therefore need to be calculated only once.
2. Only 8 inner loop operations are involved
   (a) 6 multiplications
   (b) one subtraction
   (c) one addition
3. The result is comparable to the simple parametric representation while achieving better efficiency
4. A non-origin centered ellipse can be generated by translating the origin centered ellipse.



Properties of a parametric Ellipse & circle

- The arc length *s*, curvature *k*, and tangential angle *phi* of the circle with radius *r* represented parametrically are
- *Arc length:* $s(t) = rt$ (1)
- *Curvature:* $k(t) = 1/t$ (2)
- *Tangential angle:* $\phi(t) = t/r$ (3)
- *Eccentricity* $e = \sqrt{1 - \frac{b^2}{a^2}}$ $0 \le e \le 1$ (4)
- *Perimeter (approx.)* $p = \pi\sqrt{2(a^2 + b^2)}$ (5)

Properties of a parametric curves

- *For any parametric curve:*

$$P(t) = [x(t) \quad y(t)]$$

Speed of parameterization*:*

$$v(t) = \|P'(t)\| = \|x'(t) \quad y'(t)\|$$

Eg: Consider a circle of radius r

$$P(t) = [r\cos(t) \quad r\sin(t)]$$

$$x' = -r\sin(t)$$

$$y' = r\cos(t)$$

$$v(t) = \sqrt{x'(t)^2 + y'(t)^2} = r\sqrt{\cos^2(t) + \sin^2(t)} = r$$

A parameterization is called regular iff $v(t) \neq 0$

Arc length of parametric curves

- *For any parametric curve:*

$$P(t) = \left[x(t) \quad y(t)\right]$$

With Speed of parameterization*:*

$$v(t) = \left\|P'(t)\right\| = \left\|x'(t) \quad y'(t)\right\|$$

The arc length of the curve is defined as:

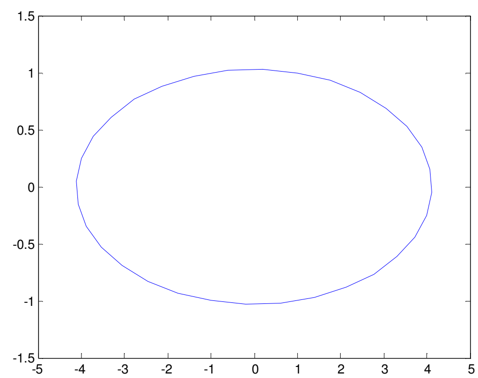
$$L(t) = \int_0^t v(t)dt = \int_0^t \sqrt{x'^2(t) + y'(t)}dt$$

For the case of circle the arc length is:

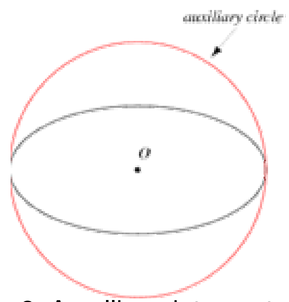
$$L(t) = \int_0^t v(t)dt = \int_0^{2\pi} [r^2 \sin^2(t) + r^2 \cos^2(t)]^{\frac{1}{2}} dt = 2\pi r$$

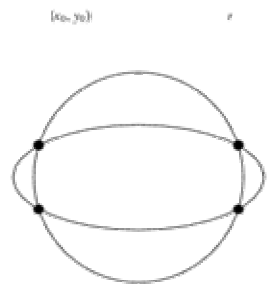
➢ Representation of an ellipse

```
%Example 4.4 (R&A)
%Ellipse a=4, b=1 inclined 30 deg. to horizontal center at (2 2)
n=33;
dt=2*pi/32;
a=4;
b=1;
sdt=sin(dt);
cdt=cos(dt);
xo=1.0;
yo=1.0;
for i=1:33
   x(i)=xo*cdt-a/b*yo*sdt;
   y(i)=xo*b/a*sdt+yo*cdt;
   xo=x(i);
   yo=y(i);
end
plot(x,y)
```



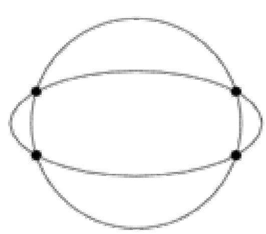
- Some Exercises
- 1. The circumcircle of an ellipse, i.e., the circle whose center concurs with that of the ellipse and whose radius is equal to the ellipse's semimajor axis.





- 2. An ellipse intersects a circle in 0, 1, 2, 3, or 4 points. The points of intersection of a circle of center (x_0,y_0) and radius r with an ellipse of semi-major and semi-minor axes a and b, respectively and center (x_e,y_e) can be determined by simultaneously solving
- (x-x_0)^2+(y-y_0)^2=r^2 (1)
- ((x-x_e)^2)/(a^2)+((y-y_e)^2)/(b^2)=1. (2)

- Some Exercises



- If (x_0,y_0)=(x_e,y_e)=(0,0), then the solution takes on the particularly simple form
- x = +/-a*sqrt((r^2-b^2)/(a^2-b^2)) (3)
- y = +/-b*sqrt((a^2-r^2)/(a^2-b^2)). (4)

Parametric Representation of a Parabola

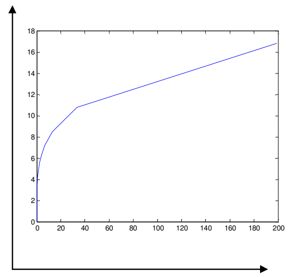
Analytical form: $y^2 = 4ax$

Parametric form:

$$x = \tan^2\phi; \quad y = \pm 2\sqrt{a\tan\phi} \qquad 0 \le \phi \le \tfrac{\pi}{2} \qquad (1)$$

An improved algorithm due to L.B.Smith (1969)

$$x = a\theta^2, \quad y = 2a\theta \qquad 0 \le \theta \le \infty$$



Where θ is the parameter that determines (x y)
Since it is an open curve we need to calculate the limits to display the curve

$$\theta_{\min} = \sqrt{\tfrac{x_{\min}}{a}}; \quad \theta_{\max} = \sqrt{\tfrac{x_{\max}}{a}} \qquad \text{based on } x$$

$$\theta_{\min} = \tfrac{y_{\min}}{2a}; \quad \theta_{\max} = \tfrac{y_{\max}}{2a} \qquad \text{based on } y$$

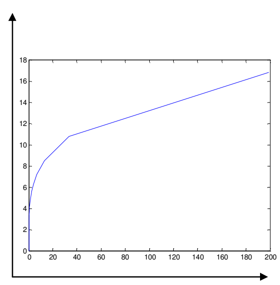
Parametric Representation of a Parabola

Starting with

$$x = a\theta^2, \quad y = 2a\theta \qquad 0 \le \theta \le \infty \qquad (1)$$

$$x_{i+1} = a\theta_i^2 + 2a\theta_i\delta\theta - a(\delta\theta)^2;$$

$$y_{i+1} = 2a\theta_i + 2a\theta_i\delta\theta$$



Substituting from (1)

$$x_{i+1} = x_i + y_i\delta\theta + a(\delta\theta)^2;$$

$$y_{i+1} = y_i + 2x_i\delta\theta \qquad \text{where } \delta\theta = \tfrac{2\pi}{(n-1)}$$

➢ Parametric Representation of a Hyperbola

Analytical form:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Parametric form: $x=\pm a\sec\theta,\quad y=\pm b\tan\theta \qquad 0\le\theta\le\frac{\pi}{2}$ (1)

Recall the identities

$$\sec(\theta+\delta\theta)=1/\cos(\theta+\delta\theta)=1/(\cos\theta\cos\delta\theta-\sin\theta\sin\delta\theta)$$

$$\tan(\theta+\delta\theta)=(\tan\theta+\tan\delta\theta)/(1-\tan\theta\tan\delta\theta)$$

Expanding sum of angles

$$x_{i+1}=\pm a\sec(\theta+\delta\theta)=\pm\frac{ab/\cos\theta}{b\cos\delta\theta-b\tan\theta\sin\delta\theta};$$

$$y_{i+1}=\pm a\tan(\theta+\delta\theta)=\pm\frac{b\tan\theta+b\tan\delta\theta}{1-\tan\theta\tan\delta\theta}$$

Substituting from (1)

$$x_{i+1}=\pm\frac{bx_i}{b\cos\delta\theta-y_i\sin\delta\theta};$$

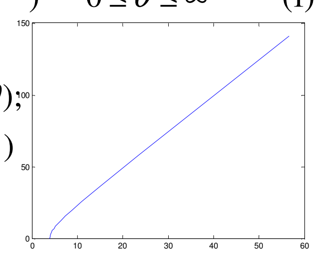
$$y_{i+1}=\pm\frac{b(y_i+b\tan\delta\theta)}{b-y_i\tan\delta\theta} \qquad \text{where } \delta\theta=\frac{2\pi}{(n-1)}$$

➢ Parametric Representation of a Hyperbola

Alternate Method

Recall the hyperbolic functions

$$x=a\cosh\theta,=(e^{\theta}+e^{-\theta}) \quad y=b\sinh\theta=(e^{\theta}-e^{-\theta}) \qquad 0\le\theta\le\infty \qquad (1)$$



Expanding sum of angles

$$x_{i+1}=a(\cosh\theta_i\cosh\delta\theta-\sinh\theta_i\sinh\delta\theta);$$

$$y_{i+1}=b(\cosh\theta_i\sinh\delta\theta+\cosh\delta\theta\sinh\theta_i)$$

Substituting from (1)

$$x_{i+1}=x_i\cosh\delta\theta-\tfrac{a}{b}y_i\sinh\delta\theta;$$

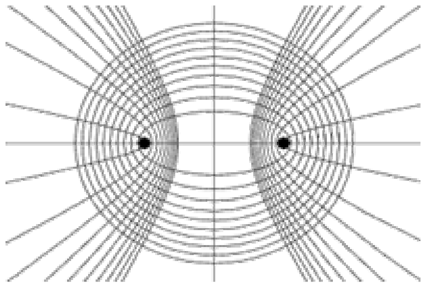
$$y_{i+1}=\tfrac{b}{a}x_i\sinh\delta\theta+y_i\cosh\delta\theta \qquad \text{where } \delta\theta=\frac{2\pi}{(n-1)}$$

The limits are

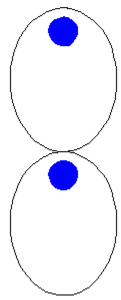
$$\theta_{\min}=\cosh^{-1}(x_{\min}/a);$$

$$\theta_{\max}=\cosh^{-1}(x_{\max}/a) \quad \text{where } \cosh^{-1}(x)=\ln(x+\sqrt{x^2-1})$$

- Parametric Representation of other plane curves
- Confocal conics



- Elliptic gears



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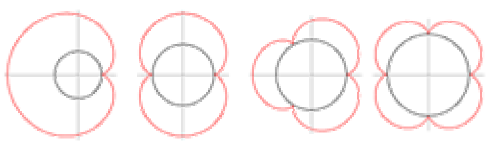
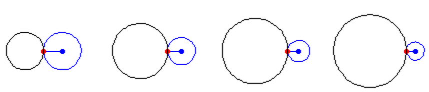
- Parametric Representation of other plane curves
- Epicycloid



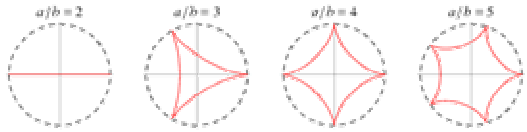


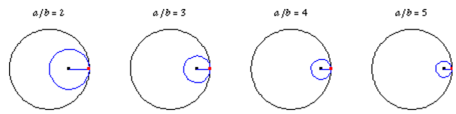
$$x = (a+b)\cos\phi - b\cos\left(\frac{a+b}{b}\phi\right)$$

$$y = (a+b)\sin\phi - b\sin\left(\frac{a+b}{b}\phi\right)$$

*Source: mathworld.**wolfram**.com*

- Parametric Representation of other plane curves
- Hypocycloid



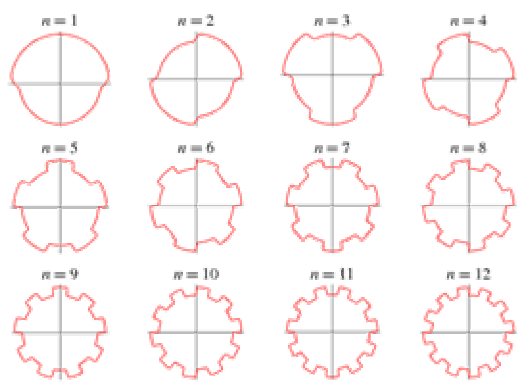


$$x = (a-b)\cos\phi - b\cos\left(\frac{a-b}{b}\phi\right)$$

$$y = (a-b)\sin\phi - b\sin\left(\frac{a-b}{b}\phi\right)$$

*Source: mathworld.**wolfram**.com*

- Parametric Representation of other plane curves
- Gear Curve



Here a=1, b=10 and n=1-12

$$x = r\cos t \qquad y = r\sin t$$

$$r = a + \frac{1}{b}\tanh[b\sin(nt)]$$

*Source: mathworld.**wolfram**.com*