Appendix B

Envelopes

The idea of an envelope plays an important role in determining solution to fully nonlinear scalar partial differential equation (see Section 2.3 for details).

B.1 ▪ Envelopes in two dimensions

Let C_m denote a family of curves in the xy-plane indexed by m, which are implicitly given by the equation $F(x,y;m)=0$. Consider the curves C_m and $C_{m+\Delta m}$ given by

$$C_m: \quad F(x,y;m)=0 \tag{B.1a}$$

$$C_{m+\Delta m}: \quad F(x,y;m+\Delta m)=0. \tag{B.1b}$$

For $\Delta m \neq 0$, the above system is same as

$$F(x,y;m)=0 \tag{B.2a}$$

$$\frac{F(x,y;m+\Delta m)-F(x,y;m)}{\Delta m}=0. \tag{B.2b}$$

Taking limit as $\Delta m \to 0$ in the equation (B.2b), we get

$$F(x,y;m)=0 \tag{B.3a}$$

$$\frac{\partial F}{\partial m}(x,y;m)=0. \tag{B.3b}$$

Definition B.1. *The envelope of the family of curves C_m given by the equation (B.1a) is defined as the set of all $(x,y)\in\mathbb{R}^2$ satisfying the system of equations (B.3) for some value of the parameter m.*

The envelope of the given family is obtained by eliminating m between the two equations in the system (B.3). Suppose we can solve for m as $m=G(x,y)$, then the envelope is given by

$$F(x,y,G(x,y))=0. \tag{B.4}$$

Remark B.2 (Properties of Envelopes). (i) The envelope touches every member of the family. That is for a point $P(x,y)$ lying on the envelope of the family of curves C_m, there exists an m_0 such that $F(x,y,m_0)=0$. This follows from the definition of envelope.

(ii) Wherever the envelope touches a particular member of the family C_m, it touches tangentially. That is, if a point $P(x,y)$ on the envelope lies on the curve C_{m_0} then both the curves have the same tangential direction at P. For, differentiating the equation C_{m_0} w.r.t. x results in

$$F_x + F_y \frac{dy}{dx} = 0.$$

On the other hand, differentiating the envelope equation (B.4) w.r.t. x yields

$$F_x + F_y \frac{dy}{dx} + \frac{\partial F}{\partial m} G_x + \frac{\partial F}{\partial m} G_y \frac{dy}{dx} = 0.$$

Since $\frac{\partial F}{\partial m}(x,y,m_0) = 0$ for (x,y) on the envelope, the last equation reduces to

$$F_x + F_y \frac{dy}{dx} = 0.$$

Thus the envelope touches each member of the family tangentially. ∎

Example B.3. The envelope of the family of straight lines

$$F(x,y;m) \equiv y - mx - \frac{a}{m} = 0$$

is the parabola $y^2 = 4ax$. ∎

B.2 ▪ Envelopes in three dimensions

Consider the following family of surfaces given by

$$S_\lambda : \quad z = G(x,y;\lambda) \tag{B.5}$$

where λ is the parameter. Keeping x,y,z fixed, differentiate $z = G(x,y;\lambda)$ w.r.t. λ to get

$$0 = G_\lambda(x,y;\lambda). \tag{B.6}$$

Suppose for a fixed λ, C_λ is the curve of intersection of the surfaces given by (B.5)-(B.6). The envelope of the family (B.5) is the union of C_λ for all λ. If we can solve for λ from equation (B.6) in terms of x and y as

$$\lambda = g(x,y), \tag{B.7}$$

then the envelope is analytically represented by the equation

$$z = G(x,y,g(x,y)) \tag{B.8}$$

which is obtained by substituting for λ in the equation (B.5).

Remark B.4. (i) The envelope E of the family of surfaces (B.5) touches every member of (B.5) along C_λ *i.e.,* the surface (B.8) touches surface (B.5) along C_λ. Suppose $(x,y,z) \in E$. Then (x,y,z) satisfies (B.5) and (B.6) for some λ. This implies that $(x,y,z) \in C_\lambda$ for some λ. Then (x,y,z) satisfies both the equations (B.5) and (B.6). In particular, $\lambda = g(x,y)$ and $(x,y,z) \in S_\lambda$.

(ii) The envelope and each member of the family S_λ have the same normal direction at every point on C_λ. In other words they have the same tangent plane at every point on C_λ. For, The normal to S_λ at (x,y,z) has direction-variables $(G_x, G_y, -1)$. The normal to E at (x,y,z) has direction-variables $(G_x+G_\lambda g_x, G_y+G_\lambda g_y, -1)=(G_x, G_y, -1)$. (since $(x,y,z)\in C_\lambda$ implies $G_\lambda = 0$). ∎

Example B.5. For $G(x,y,\lambda) \equiv ax+by+c+\lambda(a_1x+b_1y+c_1)=0$, the envelope is the point of intersection of the lines

$$ax+by+c=0,\ a_1x+b_1y+c_1=0$$

∎

Example B.6. For $G(x,y,\lambda) \equiv ax+by+cz+d+\lambda(a_1x+b_1y+c_1z+d_1)=0$, the envelope is the point of intersection of the planes

$$ax+by+cz+d=0,\ a_1x+b_1y+c_1z+d_1=0$$

∎

Exercises

2.1. Find the envelope of the family of lines given by $y=mx\pm a\sqrt{1+m^2}$.

$$F(x,y;m)\equiv y-mx\pm a\sqrt{1+m^2}=0.$$

2.2. Direction numbers of the normal to the tangent plane $(\cos\beta, \sqrt{1-\cos^2\beta-\sin^2\alpha}, \sin\alpha)$. Find the envelope of the family of planes given by

$$(\cos\beta)x+\sqrt{1-\cos^2\beta-\sin^2\alpha}y+(\sin\alpha)z=0$$

where α is a constant and β is a real parameter. (Answer: $(\tan^2\alpha)z^2=x^2+y^2$.)

2.3. Find the envelope of

$$G(x,y,z,\lambda)\equiv\lambda x+\sqrt{1-\lambda^2-\mu^2}y+\mu z=0$$

where μ is a constant and λ is a real parameter. (Answer: $(1-\mu^2)(x^2+y^2)-\mu^2z^2=0$.)

Question: Is an envelope of a family of planes always a cone? Answer is no. One can take a family of planes consisting of only one member, in which case envelope coincides with the given plane.