Chapter 1: Limits and continuity | *Section 7*

Discontinuities

What you need to know already:	***What you can learn here:***
➤ The concept and definition of continuity.	➤ The different types of discontinuity and how to recognize them.

When we are dealing with what we call a *boring* limit, we know that we have continuity. But what about the interesting limit situations? How can we analyze them to discover if we are really dealing with a discontinuity and, if so, of what type?

The key equality that defines continuity, $\lim_{x\to c} f(x) = f(c)$ is the tool to use, and we can do so in the following way.

Strategy for identifying and classifying discontinuities

To identify and classify the discontinuities of a function $f(x)$:

1. ***Identify*** the values for which something ***unusual*** occurs in the formula or definition of the function.
2. ***Evaluate the function*** at each such value.
3. ***Compute*** the left and right ***limits*** of the function at each such value.
4. If the function and the limits all exist and are equal, conclude that the function is ***continuous*** there.
5. If the equality $\lim_{x\to c} f(x) = f(c)$ does not hold, use the information about why it does not to ***classify*** it, that is, to decide what type of discontinuity exists at that value.

And how do we figure out what type of discontinuity it is?

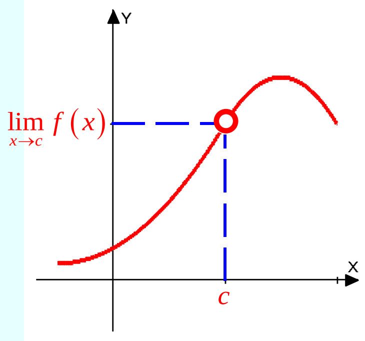
To do that, you need to know what the main types of discontinuity are and what features identify them. What follows is a list of definitions of the discontinuities that are most common and that you will see in your future work.

I will present one example of each type, since we need to develop some better methods to compute limits before we can properly analyze a larger variety of examples.

Definition

A ***single point hole*** occurs when:

- $\lim_{x \to c} f(x)$ exists, and
- $f(c)$ does not exist.

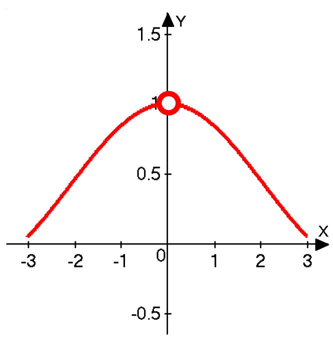


This means that the function is approaching a finite value, but does not reach it.

The resulting hole is usually represented by a small, hollow circle.

Example: $y = \dfrac{\sin x}{x}$

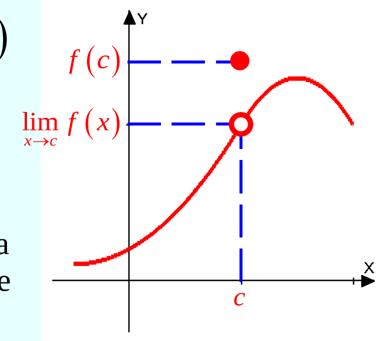
We have seen before that this function is not defined at $x = 0$, but the graphical and numerical methods suggest – and computational methods you have not seen yet confirm – that the limit exists there and equals 1. Therefore we have a single point hole there.



Definition

A ***displaced point*** occurs when:

- $\lim_{x \to c} f(x)$ and $f(c)$ exist, but …
- they are not equal.

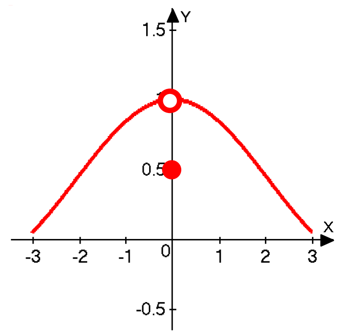


This means that the function is approaching a finite value, but the value of the function is somewhere else.

This is a fairly artificial type of discontinuity that you will see very rarely and it is described through a piecewise function.

Example:

$$\begin{cases} y = \dfrac{\sin x}{x} & if \ \ x \neq 0 \\ 0.5 & if \ \ x = 0 \end{cases}$$



This is a well defined function and, away from 0, it is the same as the previous one. But now it is defined at 0 as well, but not where we expect it. This generates a displaced point.

You can see that this is a very contrived situation: you need to know about it, especially to clarify the concept of discontinuity, but you will not see it often.

Definition

Single point holes and displaced points are also called ***removable discontinuities***, since the function can be made continuous at those values by defining:

$$f(c)=\lim_{x\to c} f(x).$$

Example: $f(x)=\dfrac{\sin x}{x}$

We can make this function continuous by letting $f(0)=1$. We can do the same with the example of a displaced point, by changing the value of the function at 0 from 0.5 to 1.

The next two types should be very familiar, but I will mention them for completeness.

Definition

A ***vertical asymptote*** occurs when:

$$\lim_{x\to c^-} f(x)=\mp\infty \quad \text{or} \quad \lim_{x\to c^+} f(x)=\mp\infty.$$

Example: $y=\dfrac{\cos x}{x}$

This function is not defined at $x=0$ and the law of balloons tells us that the limit there is infinite, so that we have a vertical asymptote there.

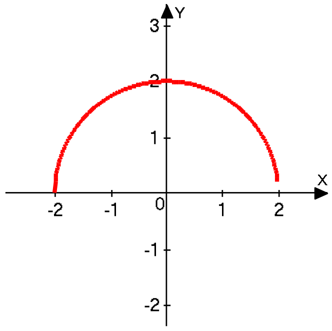
The next discontinuity is not considered such by some authors and teachers, as it does not occur inside the domain of the function, but I prefer to include it, as it is consistent with the general idea of a discontinuity indicating a break in the graph.

Definition

An ***end of domain*** discontinuity occurs when $f(x)$ is not defined on an interval to the left or to the right of the value.

Example: $y=\sqrt{4-x^2}$

The domain of this function only extends from -2 to 2. So, both values provide end of domain discontinuities, since at each of them the function exists only on one side.

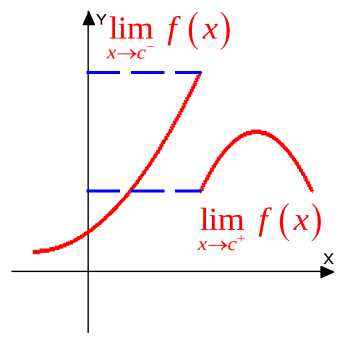


The last type of discontinuity that we shall consider may look artificial and rare, but it actually occurs often in real life, in the way that certain practical quantities are obtained.

Definition

A ***jump*** occurs when $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$ both exist, but are not equal.

In that case the graph is approaching two different, finite values from either side.



Although a jump usually occurs within a piecewise function, it can occur in functions defined by a single formula. Here s the most typical example.

Example: $f(x)=\dfrac{\sqrt{x^2}}{x}=\dfrac{|x|}{x}$

We have seen earlier that in this case $\lim\limits_{x\to 0^-} f(x)=-1$ and $\lim\limits_{x\to 0^+} f(x)=1$.

This indicates a jump. You may want to check this on your calculator.

Summary

- Discontinuities occur when continuity fails.
- The most useful and interesting aspect of a discontinuity is its classification into the specific type.

Common errors to avoid

- Do not assume continuity or discontinuity without checking the defining equality.
- Do not assume the type of discontinuity that exists at a value until you have checked its requirements.

Learning questions for Section 1.7

Memory questions:

1. Which graphical feature occurs when $\lim_{x \to a} f(x) = k$, but $f(a)$ does not exist?
2. Which graphical feature occurs when $\lim_{x \to c} f(x) \neq f(c)$, but both sides exist?
3. What is the name of the discontinuity that may occur when $\lim_{x \to c} f(x)$ exists?
4. Which two types of discontinuities are described as *removable*?
5. Which graphical feature occurs when $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, but both limits exist?
6. Which graphical feature occurs when $\lim_{x \to c} f(x) = \pm\infty$?

Review questions:

1. What does it mean to *classify a discontinuity*?

Computation questions:

For each of the following functions, identify the values at which a discontinuity may exist. Then use the graphical or numerical method to make a reasonable conjecture about the type of discontinuity. You will need more methods to compute limits in order to confirm these and other discontinuities.

1. $f(x) = \dfrac{x^2 + x + 12}{x - 3}$

2. $f(x) = \begin{cases} 2\sqrt{4 - x} & \text{if } x \leq 4 \\ \dfrac{2}{4 + x} & \text{if } x > 4 \end{cases}$

3. $f(x) = \begin{cases} \dfrac{3}{x - 5} & \text{if } x \leq 2 \\ \dfrac{5}{x + 3} & \text{if } x > 2 \end{cases}$

4. $y = \ln\left(\dfrac{x^3 - x^2}{x - 1}\right)$

5. $f(x)=\begin{cases} \dfrac{1}{2-x^2} & \text{if } x<0 \\ \dfrac{e^x}{2-x} & \text{if } 0\le x\le 1 \\ \dfrac{2}{1-x} & \text{if } x>1 \end{cases}$

6. $f(x)=\dfrac{x-\sqrt{x}}{\sqrt{x}-1}$

7. $f(x)=\begin{cases} \dfrac{5-2x}{x-5} & \text{if } x<4 \\ 3 & 4\le x\le 5 \\ 2x-8 & x>5 \end{cases}$

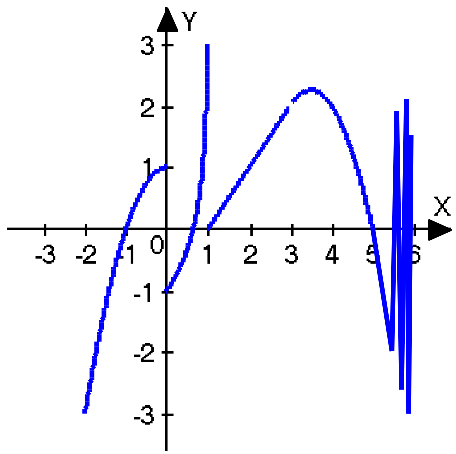
8. $f(x)=\begin{cases} e^x & \text{if } x\le 0 \\ \sin x & \text{if } x>0 \end{cases}$

9. $f(x)=\begin{cases} e^x & \text{if } x\le 0 \\ \cos x & \text{if } x>0 \end{cases}$

10. Which values of *a* and *b* make the following function continuous?

$$f(x)=\begin{cases} 2\ln x^2 & \text{if } x<-2 \\ ax+b & \text{if } -2\le x<0 \\ 8x & \text{if } x\ge 0 \end{cases}$$

11. Visually identify and tentatively classify all discontinuities of the function whose graph is shown here.



Theory questions:

1. What are the three main graphical features that we have seen occurring at points of discontinuity?

2. What two conditions tell us that a function has a single point hole at $x=c$?

3. Which of the conditions for continuity is violated in a single point hole?

4. What two conditions tell us that a function has a displaced point at $x=c$?

5. Can a function be continuous at a value where a vertical asymptote exists?

6. Do vertical asymptotes always correspond to discontinuities?

7. Is a jump considered a *removable* discontinuity?

8. When a given limit does not exist, can it correspond to a jump?

9. What kinds of function are more likely to contain a jump discontinuity?

Proof questions:

1. If $f(x)$ is a function that is continuous at x=1, what graphical features can occur at x=1 in the graph of the function $y=\frac{f(x)}{x-1}$? Identify all possible options and explain how we can distinguish them by using limits

Templated questions:

1. Identify and classify the discontinuities of any function of your choice.

What questions do you have for your instructor?