Leibnitz's Theorem:

If u and v are functions of x possessing derivatives of the n^{th} order, then

$$(uv)_n = {}^nC_0 uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_{n-1} u_{n-1} v_1 + {}^nC_n u_n v.$$

Proof: The Proof is by the principle of mathematical induction on n.

Step 1: Take n = 1

By direct differentiation, $(uv)_1 = uv_1 + u_1 v$

For $n = 2$, $(uv)_2 = u_2v + u_1v_1 + u_1v_1 + uv_2$

$$= u_2v + {}^2C_1u_1v_1 + {}^2C_2\, uv_2$$

Step 2: We assume that the theorem is true for $n = m$

$(uv)_m =$

$${}^mC_0uv_m + {}^mC_1\, u_1\, v_{m-1} + \ldots + {}^mC_{m-1}\, u_{m-1}v_1 + {}^mC_m u_m v.$$

Differentiating both sides we get

$$(uv)_{m+1} = {}^mC_0u\, v_{m+1} + {}^mC_0\, u_1\, v_m + {}^mC_1\, u_1\, v_m + {}^mC_1\, u_2\, v_{m-1} + \ldots$$

$$\ldots + {}^mC_m\, u_m\, v_1 + {}^mC_m\, u_{m+1}\, v.$$

Note: (i) $^{m}C_{r-1} + {}^{m}C_{r} = {}^{(m+1)}C_{r}$

(ii) $1 + {}^{m}C_{1} = 1+m = {}^{(m+1)}C_{1}$

(iii) $^{m}C_{m} = 1 = {}^{(m+1)}C_{m+1}$

$$(uv)_{m+1} = {}^{m}C_0 u\, v_{m+1} + ({}^{m}C_0 + {}^{m}C_1)u_1\, v_m + ({}^{m}C_1 + {}^{m}C_2)u_2\, v_{m-1} + \ldots$$

$$\ldots + ({}^{m}C_{m-1} + {}^{m}C_m)u_m v_1 + {}^{m}C_m u_{m+1} v.$$

$(uv)_{m+1} =$

$$^{m+1}C_0 uv_{m+1} + {}^{m+1}C_1\, u_1\, v_m + \ldots + {}^{m+1}C_m\, u_m v_1 + {}^{m+1}C_{m+1} u_{m+1} v.$$

Therefore the theorem is true for m + 1 and hence by the principle of mathematical induction, the theorem is true for any positive integer n.

Example: If $y = \sin(m \sin^{-1} x)$ then prove that

(i) $(1 - x^2) y_2 - xy_1 + m^2 y = 0$

(ii) $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0.$

$$y_1 = \cos(m \sin^{-1} x)\, m \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = m \cos(m \sin^{-1} x)$$

$$(1 - x^2) y_1^2 = m^2 \cos^2(m \sin^{-1} x)$$

$$= m^2 [1 - \sin^2(m \sin^{-1} x)]$$

$$= m^2 (1 - y^2).$$

Differentiating both sides we get

$$(1 - x^2)2y_1 . y_2 + y_1^2 (-2x) = m^2 (-2y . y_1)$$

$$(1 - x^2) y_2 - xy_1 + m^2 .y = 0$$

Applying Leibnitz's rule we get

$$[(1 - x^2) y_{n+2} + {}^nc_1 (-2x) . y_{n+1} + {}^nc_2 (-2) .y_n]$$

$$- [x y_{n+1} + {}^nc_1 .1. y_n] + m^2 y_n = 0$$

$$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0.$$