TAYLOR'S THEOREM

We now look at a result which allows us to compute the values of elementary functions like *sin, exp* and *log.* This theorem can be used to approximate these functions by polynomials (which are easy to compute) and provides an estimate of the error involved in the approximation.

Taylor's Theorem. *Let f be an $(n+1)$ times differentiable function on an open interval containing the points a and x. Then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

The function T_n defined by

$$T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n \quad \text{where } a_r = \frac{f^{(r)}(a)}{r!},$$

is called the *Taylor polynomial* of degree n of f at a. This can be thought of as a polynomial which approximates the function f in some interval containing a. The error in the approximation is given by the remainder term $R_n(x)$. If we can show $R_n(x) \to 0$ as $n \to \infty$ then we get a sequence of better and better approximations to f leading to a power series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which is known as the *Taylor series* for f. In general this series will converge only for certain values of x determined by the radius of convergence of the power series (see Note 17). When the Taylor polynomials converge rapidly enough, they can be used to compute approximate values of the function.

Connection with Mean Value Theorem.

When $n = 0$, Taylor's theorem reduces to the Mean Value Theorem which is itself a consequence of Rolle's theorem. A similar approach can be used to prove Taylor's theorem.

Proof of Taylor's Theorem.
The remainder term is given by

$$R_n(x) = f(x) - f(a) - f'(a)(x - a) - \frac{f''(a)}{2!}(x - a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Fix x and a. For t between x and a set

$$F(t) = f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2!}(x - t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n,$$

so that $F(a) = R_n(x)$. Then

$$
\begin{aligned}
F'(t) &= -f'(t) - f''(t)(x - t) + f'(t) - \frac{f'''(t)}{2!}(x - t)^2 + f''(t)(x - t) \\
&\quad - \cdots - \frac{f^{(n+1)}(t)}{n!}(x - t)^n + \frac{f^{(n)}(t)}{(n - 1)!}(x - t)^{n-1} \\
&= -\frac{f^{(n+1)}(t)}{n!}(x - t)^n.
\end{aligned}
$$

Now defining

$$G(t) = F(t) - \left(\frac{x - t}{x - a}\right)^{n+1} F(a),$$

we have $G(a) = 0$ and $G(x) = F(x) = 0$. Applying Rolle's theorem to the function G shows that there is a c between a and x with $G'(c) = 0$. Now

$$
\begin{aligned}
0 = G'(c) &= F'(c) + (n + 1)\frac{(x - c)^n}{(x - a)^{n+1}}F(a) \\
&= -\frac{f^{(n+1)}(c)}{n!}(x - c)^n + (n + 1)\frac{(x - c)^n}{(x - a)^{n+1}}F(a).
\end{aligned}
$$

But $F(a) = R_n(x)$ and rearranging the last equation gives

$$R_n(x) = F(a) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}. \qquad \square$$

A useful consequence of Taylor's theorem is the following generalization of the second derivative test:

To see why this is valid, note that if the first $n - 1$ derivatives all vanish at a, then by Taylor's theorem

$$f(x) - f(a) = R_{n-1}(a) = \frac{f^{(n)}(c)}{n!}(x - a)^n$$

for some c between x and a. When n is even, the term $(x - a)^n$ is always positive. If $f^{(n)}(a) > 0$ then $f^{(n)}(c) > 0$ provided x (and therefore c) is sufficiently close to a. Hence $f(x) - f(a) > 0$ for all x sufficiently close to a, so there is a local minimum at a. A similar argument shows that if $f^{(n)}(a) < 0$ then we have a local maximum. If n is odd, then $(x - a)^n$ and therefore $f(x) - f(a)$ have opposite signs for $x < a$ and $x > a$ so there is a point of inflection at a.

Maclaurin Series

Taking $a = 0$ in Taylor's theorem gives us the expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

for some number c between 0 and x. For those values of x for which $\lim_{n \to \infty} R_n(x) = 0$, we then obtain the following power series expansion for f which is known as the *Maclaurin series* of f:

(Here $f^{(0)}(0)$ is defined to be $f(0)$.)

Basic (and important) Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{R})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (x \in \mathbb{R})$$

$$(1+x)^a = 1 + ax + \frac{a(a-1)x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \binom{a}{n} x^n \qquad (|x| < 1)$$

$$\text{where } \binom{a}{n} = a(a-1)\ldots(a-(n-1))/n! \qquad (a \in \mathbb{R})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad (|x| < 1)$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad (|x| < 1)$$

Example. The Maclaurin series expansion of the exponential function is easy to find. If $f(x) = e^x$ then $f^{(n)}(x) = e^x$, so every $f^{(n)}(0)$ is 1, and

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find the values of x for which this is valid, we need to consider the remainder term (or use the Ratio Test alone; Note 17)

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \frac{x^{n+1}}{(n+1)!} e^c$$

for some c between 0 and x. It follows from the Ratio Test that the series $\sum (x^n/n!)$ converges for any x and hence the sequence $(x^n/n!)$ converges to 0. Therefore $R_n(x) \to 0$ for every $x \in \mathbb{R}$, so the Maclaurin series expansion is valid for every x.

Example. To find the Maclaurin series of the *sine* function we need to find its derivative of order n.

$$
\begin{array}{rclcrcl}
f(x) & = & \sin x & \qquad & f(0) & = & 0 \\
f'(x) & = & \cos x & & f'(0) & = & 1 \\
f''(x) & = & -\sin x & & f''(0) & = & 0 \\
f'''(x) & = & -\cos x & & f'''(0) & = & -1 \\
f^{(4)}(x) & = & \sin x & & f^{(4)}(0) & = & 0
\end{array}
$$

It follows that $f^{(n)}(0) = 0$ if n is even, and alternates as $1, -1, 1, -1, \ldots$ for $n = 1, 3, 5, 7, \ldots$. Hence the Maclaurin series expansion is

$$
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots
$$

or
$$
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.
$$

To find values of x for which this is valid, we need to consider the remainder term (or use the Ratio Test) which is given by

$$
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.
$$

Now for each n, $f^{(n+1)}(c)$ is given by $\pm \sin c$ or $\pm \cos c$. The values of the sine and cosine functions always lie between -1 and 1, so

$$
\frac{-x^{n+1}}{(n+1)!} \leq R_n(x) \leq \frac{x^{n+1}}{(n+1)!},
$$

and since $x^{(n+1)}/(n+1)! \to 0$, we get $R_n(x) \to 0$ by the Squeeze Rule. This shows that the Maclaurin series expansion is valid for all $x \in \mathbb{R}$.

ABSTRACT

Content *definition, proof of Taylor's Theorem, n^{th} derivative test for stationary points, Maclaurin series, basic Maclaurin series*

In this Note, we look at a Theorem which plays a key role in mathematical analysis and in many other areas such as numerical analysis. The well-known derivative test for maxima and minima of functions is generalised and Maclaurin Series are introduced.