TAYLOR and MACLAURIN SERIES

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TAYLOR SERIES

Recall our discussion of the power series, the power series will converge absolutely for every value of x in the interval of convergence. Also the sum of a power series is a continuous function with derivatives of all orders within this interval. So the question is this: If a function f (x) has derivatives of all orders on an interval I, can it be expressed as a power series on I? Furthermore, what would be the coefficients of this series? Let us see if we can determine the coefficients.

$$\text{Let } f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + \cdots$$

and let this power series have a positive radius of convergence. Now do repeated term-by-term differentiation within the interval of convergence I.

$f'(x) = a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + \ldots + na_n(x - a)^{n-1} + \ldots$

$f'' = 2a_2 + (2)(3)a_3(x - a) + (3)(4)a_4(x - a)^2 + \ldots + n(n - 1)a_n(x - a)^{n-2} + \ldots$

$f''' = (2)(3)a_3 + (2)(3)(4)a_4(x - a) + \ldots + n(n - 1)(n - 2)a_n(x - a)^{n-3} + \ldots$

$f^n = n!a_n$ + a sum of terms with $(x - a)$ as a factor.

All of these equations hold for $x = a$, therefore, $f'(a) = a_1$, $f''(a) = (1)(2)a_2$, $f'''(a) = (1)(2)(3)a_3$, …, $f^n(a) = n!a_n$. Do you notice a pattern? If there is a power series for f (x) that converges on I, then the coefficients are of the form

$$a_n = \frac{f^n(a)}{n!}$$

and

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

The above series is called the Taylor series generated by f at x = a. If x = 0, then the series looks like this:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

and this is called the Maclaurin series generated by f at x = 0.

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Saylor URL: http://www.saylor.org/courses/ma102/

EXAMPLE 1: Find the Taylor series about x = -1 for f (x) = 1/x. Express your answer in sigma notation.

SOLUTION:

$f(x) = x^{-1}$	$f(-1) = -1$
$f' = -x^{-2}$	$f'(-1) = -1$
$f'' = 2x^{-3}$	$f''(-1) = -2$
$f''' = -6x^{-4}$	$f'''(-1) = -6$
$f'''' = 24x^{-5}$	$f''''(-1) = -24$

$$f(x) = -1 - 1(x+1) - \frac{2}{2!}(x+1)^2 - \frac{6}{3!}(x+1)^3 - \frac{24}{4!}(x+1)^4 - \dots$$

$$= \sum_{n=0}^{\infty} -1(x+1)^n$$

EXAMPLE 2: Find the Maclaurin series for f (x) = sin π x. Express your answer in sigma notation.

SOLUTION:

$f(x) = \sin \pi x$	$f(0) = 0$
$f'(x) = \pi \cos \pi x$	$f'(0) = \pi$
$f'' = -\pi^2 \sin \pi x$	$f''(0) = 0$
$f''' = -\pi^3 \cos \pi x$	$f'''(0) = -\pi^3$
$f'''' = \pi^4 \sin \pi x$	$f''''(0) = 0$
$f^5 = \pi^5 \cos \pi x$	$f^5(0) = \pi^5$

$$f(x) = 0 + \pi x + \frac{0\,x^2}{2!} - \frac{\pi^3}{3!}x^3 + \frac{0}{4!}x^4 + \frac{\pi^5}{5!}x^5 - \dots$$

$$= \pi x - \frac{\pi^3}{6}x^3 + \frac{\pi^5}{120}x^5 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$$

EXAMPLE 3: Find the Maclaurin series for f (x) = x e ^{x}. Express your answer in sigma notation.

SOLUTION:

$f(x) = x e^x$	$f(0) = 0$
$f' = e^x + x e^x$	$f'(0) = 1 + 0 = 1$
$f'' = e^x + e^x + x e^x$	$f''(0) = 1 + 1 + 0 = 2$
$f''' = e^x + e^x + e^x + x e^x$	$f'''(0) = 1 + 1 + 1 + 0 = 3$

f '''' = $e^x + e^x + e^x + e^x + x e^x$ f ''''(0) = 1 + 1 + 1 + 1 + 0 = 4

$$f(x) = 0 + 1x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 + \cdots$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

The degrees of the x terms starts at one, whereas the denominator starts at 0!.

TAYLOR POLYNOMIALS

What is the difference between a Taylor series and a Taylor polynomial? The Taylor series is an infinite series, whereas a Taylor polynomial is a polynomial of degree n and has a finite number of terms. The form of a Taylor polynomial of degree n for a function f (x) at x = a is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

EXAMPLE 4: Find the third Taylor polynomial for f (x) = $\tan^{-1}$ x at x = 1.

SOLUTION: The third Taylor polynomial is when n = 3, so first I will find the first three derivatives of f (x) and evaluate them at x = 1.

f (x) = $\tan^{-1}$ x $\qquad f(1) = \frac{\pi}{4}$

$$f' = \frac{1}{1+x^2} \qquad f'(1) = \frac{1}{2}$$

$$f'' = -2x(1+x^2)^{-2} \qquad f''(1) = -\frac{2}{4} = -\frac{1}{2}$$

$$f''' = \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} \qquad f'''(1) = -\frac{2}{4} + \frac{8}{8} = \frac{1}{2}$$

$$P_3(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{\frac{1}{2}}{2!}(x-1)^2 + \frac{\frac{1}{2}}{3!}(x-1)^3$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$$

EXAMPLE 5: Find the fourth Maclaurin polynomial for f (x) = sin 2x.

SOLUTION:

f (x) = sin 2x	f (0) = 0
f ' = 2cos 2x	f '(0) = 2
f " = -4sin 2x	f "(0) = 0
f "' = -8cos 2x	f "'(0) = -8
f "" = 16sin 2x	f ""(0) = 0

$$P_4(x) = 0 + 2x - \frac{0}{2!}x^2 - \frac{8}{3!}x^3 + \frac{0}{4!}x^4 = 2x - \frac{4}{3}x^3$$

EXAMPLE 6: Find the fourth Taylor polynomial for f (x) = ln x at x = 1.

SOLUTION:

f (x) = ln x	f (1) = 0
f ' = x^{-1}	f ' (1) = 1
f " = $-x^{-2}$	f " (1) = -1
f "' = $2x^{-3}$	f "' (1) = 2
f "" = $-6x^{-4}$	f "" (1) = -6

$$P_4(x) = 0 + (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

EXAMPLE 7 Find the Taylor series for f (x) = $3x^5 - x^4 + 2x^3 + x^2 - 2$ at x = -1.

SOLUTION:

f (x) = $3x^5 - x^4 + 2x^3 + x^2 - 2$	f (-1) = -3 - 1 - 2 + 1 - 2 = -7
f '(x) = $15x^4 - 4x^3 + 6x^2 + 2x$	f '(-1) = 15 + 4 + 6 - 2 = 23
f "(x) = $60x^3 - 12x^2 + 12x + 2$	f "(-1) = -60 - 12 -12 + 2 = -82
f "'(x) = $180x^2 - 24x + 12$	f "'(-1) = 180 + 24 + 12 = 216
f ""(x) = 360x - 24	f ""(-1) = -360 - 24 = -384
f ""'(x) = 360	

$$f(x) = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3$$

$$- \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5 = -7 + 23(x+1) - 41(x+1)^2$$

$$+ 36(x+1)^4 - 16(x+1)^4 + 3(x+1)^5$$

Now that I have introduced the topic of power, Taylor, and Maclaurin series, we will now be ready to determine Taylor or Maclaurin series for specific functions. In the next set of supplemental notes, we will discuss how to use these series to help us determine the value of a non-elementary integrals and limits of indeterminate forms.