# Lengths of Plane Curves

For a general curve in a two-dimensional plane it is not clear exactly how to measure its length. In everyday physical situations one can place a string on top of the curve, and then measure the length of the string when it is straightened out, noting that the length of the string is the same whether it is wound up or not. Unfortunately, we have no means of running a string over an arbtirary curve, one that we might not even be able to sketch. Instead, we need to use the notion of approximation, and use a limit to make the approximation as accurate as we would like. The simplest means of approximating a curve is using straight line segments. As we increase the number of segments, they begin to hang closer and closer to the curve, and in the limit that the number of segments approaches infinity, we find the exact length of the curve.

Let $L$ denote the length of a curve $C$, and correspondingly let $ds$ represent the length of a segment of infinitessimal length of the curve. If we sum up the lengths of all of these infinitessimal pieces, then we will find the entire length of the curve. This is conveniently expressed in integral notation as

$$L = \int_C ds$$

which can be interpreted as saying the length of a curve is given the sum of the length of infinitessimal segments of the curve. Although this notation conveniently represents the problem, it does not tell us how to find its solution. What we need to do is represent the differential $ds$ in terms of the variables $x$ and $y$. If we draw a line segment between two points of the curve, the length of the segment will be the hypotenuse of a triangle with sides $\Delta x$ and $\Delta y$ which represent the function's change in the $x$ and $y$ coordinates. If we let $\Delta s$ denote the length of the hypotenuse, using the Pythagorean theorem we find

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

or

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Next we note that as the lengths of $\Delta x$ and $\Delta y$ tend to 0, the length of the hypotenuse of this triangle becomes an increasingly accurate estimate of the length of the change of the curve. Thus, when we consider infinitessimal changes in $x$ and $y$, we find that

$$ds = \sqrt{dx^2 + dy^2}$$

which gives us a means of representing $ds$ in terms of known quantities. However, we are still not yet finished. We now need a means of representing the differentials $dx$ and $dy$. The way we represent these quantities depends on the form the curve is represented in. If we have a differentiable function $y(x)$, then we can write

$$dy = \frac{dy}{dx} dx$$

and

$$ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

In order to find the length of the curve we would simply integrate over the appropriate limits in $x$, so if we want to find the length of the function as $x$ varies from $a$ to $b$, we would calculate

$$L = \int_C ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Similarly, if we had a differentiable function $x(y)$ then we would represent

$$dx = \frac{dx}{dy} dy$$

and

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + dy^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Then if we wanted to find the length as $y$ varied from $c$ to $d$, we would simply calculate

$$L = \int_C ds = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Finally, it is worth noting that the number of curves we can represent as functions $y(x)$ or $x(y)$ is rather limited. By looking to parametrized curves we could greatly increase the number of curves for which we can calculate the length.

Consider the following examples.

**Example 1** Find the length of the curve $y = x^{3/2}$ from $0 \leq x \leq 2$.
**Solution** Here we have $y$ as a function of $x$, so we want to represent $ds$ in terms of $x$. We find that

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

so

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

Now evaluating the appropriate integral

$$L = \int_0^2 \sqrt{1 + \frac{9}{4} x} dx$$

which we solve through substitution, letting

$$u = 1 + \frac{9}{4} x$$

and

$$du = \frac{9}{4} dx \quad \text{or} \quad \frac{4du}{9} = dx$$

and so $u(0) = 1$ and $u(2) = 1 + \frac{9}{2} = \frac{11}{2}$. Thus,

$$L = \int_0^2 \sqrt{1 + \frac{9}{4}x}dx = \frac{4}{9}\int_1^{11/2} \sqrt{u}du = \frac{8}{27}u^{3/2}\Big|_1^{11/2} = \frac{8}{27} \cdot \left[\left(\frac{11}{2}\right)^{3/2} - 1\right] \approx 3.5255$$

**Example 2** Find the length of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.
**Solution** When we find

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \cdot \left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

we notice that this derivative does not exist for $x = 0$. In fact, this function grows without bound as $x \to 0$, which means that it is not Riemann integrable (only bounded functions are Riemann integrable). Thus, we cannot directly calculate the length of this curve. However, can alternative represent $x$ as a function of $y$, and find

$$\begin{aligned} y &= \left(\frac{x}{2}\right)^{2/3} \\ y^{3/2} &= \frac{x}{2} \\ x &= 2y^{3/2} \end{aligned}$$

which we have previously seen is differentiable. When $x = 0$ we have $y = 0$, and when $x = 2$ we have $y = 1$, so our limits of integration will be for $y$ from 0 to 1. Calculating the derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

and

$$\left(\frac{dx}{dy}\right)^2 = 9y$$

Thus,

$$L = \int_0^1 \sqrt{1 + 9y}dy = \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2}\Big|_0^1 = \frac{2}{27}(10\sqrt{10} - 1) \approx 2.27$$

**Example 3** Find the length of $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$.
**Solution** We begin by finding the derivative

$$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$$

and

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{4}(y^4 - 2 + y^{-4})$$

Thus,

$$\begin{aligned} L &= \int_2^3 \sqrt{1 + \frac{1}{4}(y^4 - 2 + y^{-4})}dy = \int_2^3 \sqrt{\frac{1}{4}(y^4 + 2 + y^{-4})}dy = \frac{1}{2}\int_2^3 \sqrt{(y^2 + y^{-2})^2}dy \\ &= \frac{1}{2}\int_2^3 (y^2 + y^{-2})dy = \frac{1}{2}\left[\frac{y^3}{3} - \frac{1}{y}\right|_2^3 = \frac{1}{2}\left[\left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right] = \frac{13}{4} \end{aligned}$$

The key to solving the above problem was the fact that

$$1 + \left(\frac{dx}{dy}\right)^2$$

was a perfect square, so it canceled out with the square root. However, most curves do not work out so nicely. Unfortunately, we cannot evaluate the vast majority of integrals that arise in finding arc length, and most of them cannot be evaluated by hand. The following is an example using an integral we cannot calculate.

**Example 4** Find the circumference of the circle $x^2 + y^2 = a^2$.
**Solution** This curve is not given by a single function, but two functions. In order to find the circumference we could calculate the length of the cuves given by each function, but alternatively we can simply exploit the symmetry of the two functions - they both have the same length. If we find the length of one of the curves, we simply double it to find the circumference of the circle. Here we have the choice of calculating the length as an integral in $x$ or $y$. First, let us write

$$y = \sqrt{a^2 - x^2}$$

so

$$\frac{dy}{dx} = -\frac{2x}{\sqrt{a^2 - x^2}}$$

and

$$\left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{a^2 - x^2}$$

Finally,

$$L = \int_{-a}^{a} \sqrt{1 + \frac{4x^2}{a^2 - x^2}} dx$$

which we do not how to evaluate, but we know the result is $L = \pi$.