10.9 Polar Equations of Conics

What you should learn

- Define conics in terms of eccentricity.
- Write and graph equations of conics in polar form.
- Use equations of conics in polar form to model real-life problems.

Why you should learn it

The orbits of planets and satellites can be modeled with polar equations. For instance, in Exercise 58 on page 798, a polar equation is used to model the orbit of a satellite.

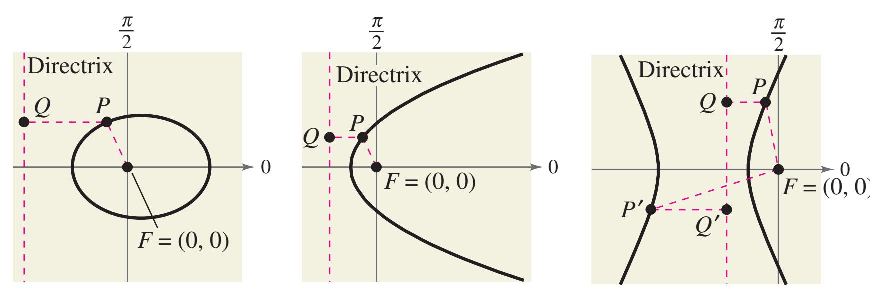
Alternative Definition of Conic

In Sections 10.3 and 10.4, you learned that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at their *centers*. As it happens, there are many important applications of conics in which it is more convenient to use one of the *foci* as the origin. In this section, you will learn that polar equations of conics take simple forms if one of the foci lies at the pole.

To begin, consider the following alternative definition of conic that uses the concept of eccentricity.

> **Alternative Definition of Conic**
>
> The locus of a point in the plane that moves so that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a **conic.** The constant ratio is the **eccentricity** of the conic and is denoted by e. Moreover, the conic is an **ellipse** if $e < 1$, a **parabola** if $e = 1$, and a **hyperbola** if $e > 1$. (See Figure 10.77.)

In Figure 10.77, note that for each type of conic, the focus is at the pole.



Ellipse: $0 < e < 1$, $\frac{PF}{PQ} < 1$

Parabola: $e = 1$, $\frac{PF}{PQ} = 1$

Hyperbola $e > 1$, $\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

FIGURE 10.77

Polar Equations of Conics

The benefit of locating a focus of a conic at the pole is that the equation of the conic takes on a simpler form. For a proof of the polar equations of conics, see Proofs in Mathematics on page 808.

> **Polar Equations of Conics**
>
> The graph of a polar equation of the form
>
> **1.** $r = \dfrac{ep}{1 \pm e\cos\theta}$ or **2.** $r = \dfrac{ep}{1 \pm e\sin\theta}$
>
> is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

Equations of the form

$$r = \frac{ep}{1 \pm e \cos \theta} = g(\cos \theta)$$ Vertical directrix

correspond to conics with a vertical directrix and symmetry with respect to the polar axis. Equations of the form

$$r = \frac{ep}{1 \pm e \sin \theta} = g(\sin \theta)$$ Horizontal directrix

correspond to conics with a horizontal directrix and symmetry with respect to the line $\theta = \pi/2$. Moreover, the converse is also true—that is, any conic with a focus at the pole and having a horizontal or vertical directrix can be represented by one of the given equations.

Example 1 Identifying a Conic from Its Equation

Identify the type of conic represented by the equation $r = \dfrac{15}{3 - 2 \cos \theta}$.

Algebraic Solution

To identify the type of conic, rewrite the equation in the form $r = (ep)/(1 \pm e \cos \theta)$.

$$r = \frac{15}{3 - 2 \cos \theta}$$ Write original equation.

$$= \frac{5}{1 - (2/3) \cos \theta}$$ Divide numerator and denominator by 3.

Because $e = \frac{2}{3} < 1$, you can conclude that the graph is an ellipse.

CHECKPOINT Now try Exercise 11.

Graphical Solution

You can start sketching the graph by plotting points from $\theta = 0$ to $\theta = \pi$. Because the equation is of the form $r = g(\cos \theta)$, the graph of r is symmetric with respect to the polar axis. So, you can complete the sketch, as shown in Figure 10.78. From this, you can conclude that the graph is an ellipse.

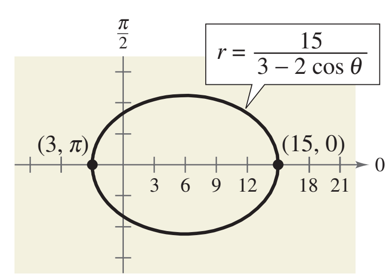


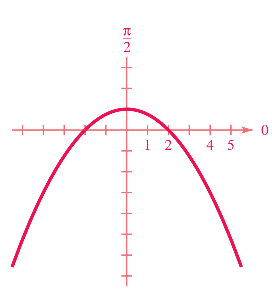
FIGURE 10.78

Additional Example
Identify the conic and sketch its graph.

$r = \dfrac{4}{2 + 2 \sin \theta}$

Solution

Parabola



For the ellipse in Figure 10.78, the major axis is horizontal and the vertices lie at $(15, 0)$ and $(3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the *minor* axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$\begin{aligned} b^2 &= a^2 - c^2 \\ &= a^2 - (ea)^2 \\ &= a^2(1 - e^2). \end{aligned}$$ Ellipse

Because $e = \frac{2}{3}$, you have $b^2 = 9^2\left[1 - \left(\frac{2}{3}\right)^2\right] = 45$, which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= (ea)^2 - a^2 \\ &= a^2(e^2 - 1). \end{aligned}$$ Hyperbola

Example 2 Sketching a Conic from Its Polar Equation

Identify the conic $r = \dfrac{32}{3 + 5\sin\theta}$ and sketch its graph.

Solution

Dividing the numerator and denominator by 3, you have

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$, and the vertices occur at $(4, \pi/2)$ and $(-16, 3\pi/2)$. Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

So, $b = 8$. Finally, you can use a and b to determine that the asymptotes of the hyperbola are $y = 10 \pm \frac{3}{4}x$. The graph is shown in Figure 10.79.

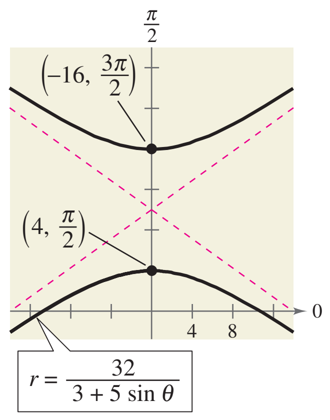


FIGURE 10.79

CHECKPOINT Now try Exercise 19.

Technology

Use a graphing utility set in *polar* mode to verify the four orientations shown at the right. Remember that e must be positive, but p can be positive or negative.

In the next example, you are asked to find a polar equation of a specified conic. To do this, let p be the distance between the pole and the directrix.

1. *Horizontal directrix above the pole:* $r = \dfrac{ep}{1 + e\sin\theta}$
2. *Horizontal directrix below the pole:* $r = \dfrac{ep}{1 - e\sin\theta}$
3. *Vertical directrix to the right of the pole:* $r = \dfrac{ep}{1 + e\cos\theta}$
4. *Vertical directrix to the left of the pole:* $r = \dfrac{ep}{1 - e\cos\theta}$

Example 3 Finding the Polar Equation of a Conic

Find the polar equation of the parabola whose focus is the pole and whose directrix is the line $y = 3$.

Solution

From Figure 10.80, you can see that the directrix is horizontal and above the pole, so you can choose an equation of the form

$$r = \frac{ep}{1 + e\sin\theta}.$$

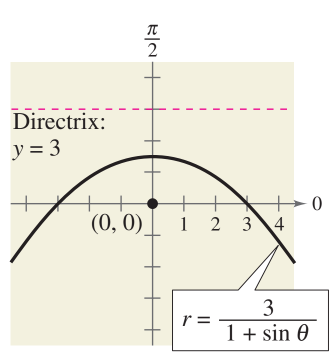


FIGURE 10.80

Moreover, because the eccentricity of a parabola is $e = 1$ and the distance between the pole and the directrix is $p = 3$, you have the equation

$$r = \frac{3}{1 + \sin\theta}.$$

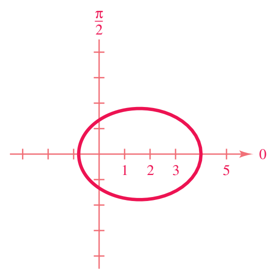
CHECKPOINT Now try Exercise 33.

Activities

1. Identify the conic and sketch its graph.

   $r = \dfrac{4}{3 - 2\cos\theta}$

   Answer: Ellipse



2. Find a polar equation of the parabola with focus at the pole and directrix $y = 2$.

   Answer: $r = \dfrac{2}{1 + \sin\theta}$

Applications

Kepler's Laws (listed below), named after the German astronomer Johannes Kepler (1571–1630), can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun at one focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period (the time it takes for a planet to orbit the sun) is proportional to the cube of the mean distance between the planet and the sun.

Although Kepler simply stated these laws on the basis of observation, they were later validated by Isaac Newton (1642–1727). In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is illustrated in the next example, which involves the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

If you use Earth as a reference with a period of 1 year and a distance of 1 astronomical unit (an *astronomical unit* is defined as the mean distance between Earth and the sun, or about 93 million miles), the proportionality constant in Kepler's third law is 1. For example, because Mars has a mean distance to the sun of $d = 1.524$ astronomical units, its period P is given by $d^3 = P^2$. So, the period of Mars is $P \approx 1.88$ years.

Example 4 Halley's Comet

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. Find a polar equation for the orbit. How close does Halley's comet come to the sun?

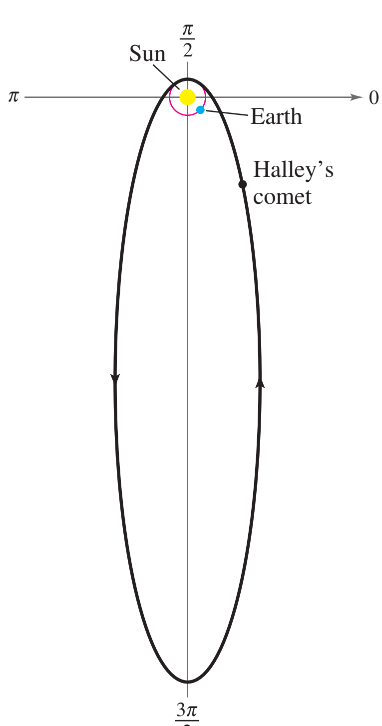


FIGURE 10.81

Solution

Using a vertical axis, as shown in Figure 10.81, choose an equation of the form $r = ep/(1 + e\sin\theta)$. Because the vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices. That is,

$$2a = \frac{0.967p}{1 + 0.967} + \frac{0.967p}{1 - 0.967} \approx 29.79p \approx 35.88.$$

So, $p \approx 1.204$ and $ep \approx (0.967)(1.204) \approx 1.164$. Using this value of ep in the equation, you have

$$r = \frac{1.164}{1 + 0.967\sin\theta}$$

where r is measured in astronomical units. To find the closest point to the sun (the focus), substitute $\theta = \pi/2$ in this equation to obtain

$$r = \frac{1.164}{1 + 0.967\sin(\pi/2)} \approx 0.59 \text{ astronomical unit} \approx 55{,}000{,}000 \text{ miles.}$$

CHECKPOINT Now try Exercise 57.

10.9 Exercises

VOCABULARY CHECK:

In Exercises 1–3, fill in the blanks.

1. The locus of a point in the plane that moves so that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a ________.

2. The constant ratio is the ________ of the conic and is denoted by ________.

3. An equation of the form $r = \dfrac{ep}{1 + e\cos\theta}$ has a ________ directrix to the ________ of the pole.

4. Match the conic with its eccentricity.

(a) $e < 1$ (b) $e = 1$ (c) $e > 1$

(i) parabola (ii) hyperbola (iii) ellipse

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

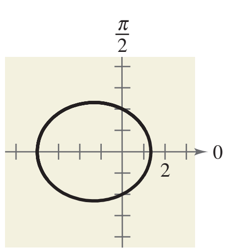
In Exercises 1–4, write the polar equation of the conic for $e = 1$, $e = 0.5$, and $e = 1.5$. Identify the conic for each equation. Verify your answers with a graphing utility.

1. $r = \dfrac{4e}{1 + e\cos\theta}$ **2.** $r = \dfrac{4e}{1 - e\cos\theta}$

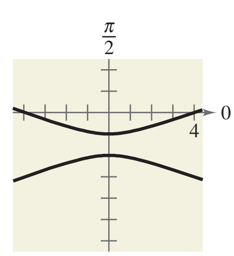
3. $r = \dfrac{4e}{1 - e\sin\theta}$ **4.** $r = \dfrac{4e}{1 + e\sin\theta}$

In Exercises 5–10, match the polar equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

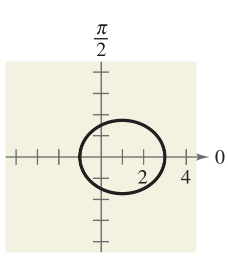
(a)



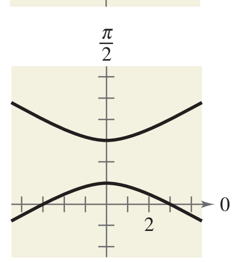
(b)



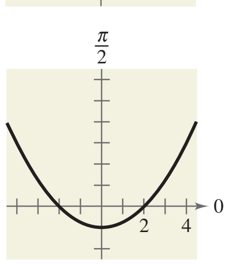
(c)



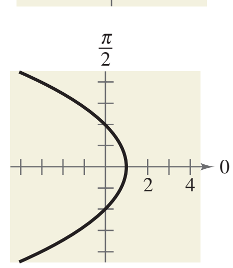
(d)



(e)



(f)



5. $r = \dfrac{2}{1 + \cos\theta}$ **6.** $r = \dfrac{3}{2 - \cos\theta}$

7. $r = \dfrac{3}{1 + 2\sin\theta}$ **8.** $r = \dfrac{2}{1 - \sin\theta}$

9. $r = \dfrac{4}{2 + \cos\theta}$ **10.** $r = \dfrac{4}{1 - 3\sin\theta}$

In Exercises 11–24, identify the conic and sketch its graph.

11. $r = \dfrac{2}{1 - \cos\theta}$ **12.** $r = \dfrac{3}{1 + \sin\theta}$

13. $r = \dfrac{5}{1 + \sin\theta}$ **14.** $r = \dfrac{6}{1 + \cos\theta}$

15. $r = \dfrac{2}{2 - \cos\theta}$ **16.** $r = \dfrac{3}{3 + \sin\theta}$

17. $r = \dfrac{6}{2 + \sin\theta}$ **18.** $r = \dfrac{9}{3 - 2\cos\theta}$

19. $r = \dfrac{3}{2 + 4\sin\theta}$ **20.** $r = \dfrac{5}{-1 + 2\cos\theta}$

21. $r = \dfrac{3}{2 - 6\cos\theta}$ **22.** $r = \dfrac{3}{2 + 6\sin\theta}$

23. $r = \dfrac{4}{2 - \cos\theta}$ **24.** $r = \dfrac{2}{2 + 3\sin\theta}$

In Exercises 25–28, use a graphing utility to graph the polar equation. Identify the graph.

25. $r = \dfrac{-1}{1 - \sin\theta}$ **26.** $r = \dfrac{-5}{2 + 4\sin\theta}$

27. $r = \dfrac{3}{-4 + 2\cos\theta}$ **28.** $r = \dfrac{4}{1 - 2\cos\theta}$

In Exercises 29–32, use a graphing utility to graph the rotated conic.

29. $r = \dfrac{2}{1 - \cos(\theta - \pi/4)}$ (See Exercise 11.)

30. $r = \dfrac{3}{3 + \sin(\theta - \pi/3)}$ (See Exercise 16.)

31. $r = \dfrac{6}{2 + \sin(\theta + \pi/6)}$ (See Exercise 17.)

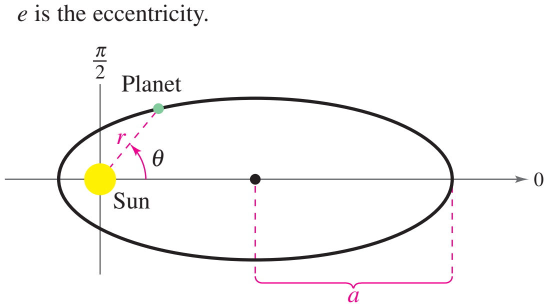
32. $r = \dfrac{5}{-1 + 2\cos(\theta + 2\pi/3)}$ (See Exercise 20.)

In Exercises 33–48, find a polar equation of the conic with its focus at the pole.

	Conic	*Eccentricity*	*Directrix*
33.	Parabola	$e = 1$	$x = -1$
34.	Parabola	$e = 1$	$y = -2$
35.	Ellipse	$e = \frac{1}{2}$	$y = 1$
36.	Ellipse	$e = \frac{3}{4}$	$y = -3$
37.	Hyperbola	$e = 2$	$x = 1$
38.	Hyperbola	$e = \frac{3}{2}$	$x = -1$

	Conic	*Vertex or Vertices*
39.	Parabola	$(1, -\pi/2)$
40.	Parabola	$(6, 0)$
41.	Parabola	$(5, \pi)$
42.	Parabola	$(10, \pi/2)$
43.	Ellipse	$(2, 0), (10, \pi)$
44.	Ellipse	$(2, \pi/2), (4, 3\pi/2)$
45.	Ellipse	$(20, 0), (4, \pi)$
46.	Hyperbola	$(2, 0), (8, 0)$
47.	Hyperbola	$(1, 3\pi/2), (9, 3\pi/2)$
48.	Hyperbola	$(4, \pi/2), (1, \pi/2)$

49. ***Planetary Motion*** The planets travel in elliptical orbits with the sun at one focus. Assume that the focus is at the pole, the major axis lies on the polar axis, and the length of the major axis is $2a$ (see figure). Show that the polar equation of the orbit is $r = a(1 - e^2)/(1 - e\cos\theta)$ where e is the eccentricity.



50. ***Planetary Motion*** Use the result of Exercise 49 to show that the minimum distance (*perihelion distance*) from the sun to the planet is $r = a(1 - e)$ and the maximum distance (*aphelion distance*) is $r = a(1 + e)$.

Planetary Motion **In Exercises 51–56, use the results of Exercises 49 and 50 to find the polar equation of the planet's orbit and the perihelion and aphelion distances.**

51. Earth $a = 95.956 \times 10^6$ miles, $e = 0.0167$

52. Saturn $a = 1.427 \times 10^9$ kilometers, $e = 0.0542$

53. Venus $a = 108.209 \times 10^6$ kilometers, $e = 0.0068$

54. Mercury $a = 35.98 \times 10^6$ miles, $e = 0.2056$

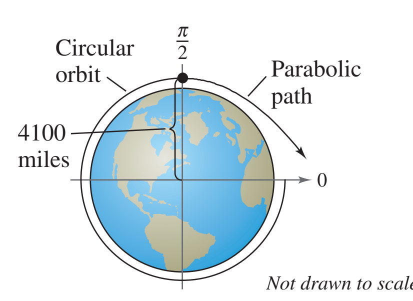
55. Mars $a = 141.63 \times 10^6$ miles, $e = 0.0934$

56. Jupiter $a = 778.41 \times 10^6$ kilometers, $e = 0.0484$

57. ***Astronomy*** The comet Encke has an elliptical orbit with an eccentricity of $e \approx 0.847$. The length of the major axis of the orbit is approximately 4.42 astronomical units. Find a polar equation for the orbit. How close does the comet come to the sun?

Model It

58. ***Satellite Tracking*** A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. If this velocity is multiplied by $\sqrt{2}$, the satellite will have the minimum velocity necessary to escape Earth's gravity and it will follow a parabolic path with the center of Earth as the focus (see figure).



(a) Find a polar equation of the parabolic path of the satellite (assume the radius of Earth is 4000 miles).

(b) Use a graphing utility to graph the equation you found in part (a).

(c) Find the distance between the surface of the Earth and the satellite when $\theta = 30°$.

(d) Find the distance between the surface of Earth and the satellite when $\theta = 60°$.

Synthesis

True or False? **In Exercises 59–61, determine whether the statement is true or false. Justify your answer.**

59. For a given value of $e > 1$ over the interval $\theta = 0$ to $\theta = 2\pi$, the graph of

$$r = \frac{ex}{1 - e\cos\theta}$$

is the same as the graph of

$$r = \frac{e(-x)}{1 + e\cos\theta}.$$

60. The graph of

$$r = \frac{4}{-3 - 3\sin\theta}$$

has a horizontal directrix above the pole.

61. The conic represented by the following equation is an ellipse.

$$r^2 = \frac{16}{9 - 4\cos\left(\theta + \dfrac{\pi}{4}\right)}$$

62. ***Writing*** In your own words, define the term *eccentricity* and explain how it can be used to classify conics.

63. Show that the polar equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$

64. Show that the polar equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.$$

In Exercises 65–70, use the results of Exercises 63 and 64 to write the polar form of the equation of the conic.

65. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$

66. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

67. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

68. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

69. Hyperbola One focus: $(5, \pi/2)$
Vertices: $(4, \pi/2), (4, -\pi/2)$

70. Ellipse One focus: $(4, 0)$
Vertices: $(5, 0), (5, \pi)$

71. ***Exploration*** Consider the polar equation

$$r = \frac{4}{1 - 0.4\cos\theta}.$$

(a) Identify the conic without graphing the equation.

(b) Without graphing the following polar equations, describe how each differs from the given polar equation.

$$r_1 = \frac{4}{1 + 0.4\cos\theta}, \quad r_2 = \frac{4}{1 - 0.4\sin\theta}$$

(c) Use a graphing utility to verify your results in part (b).

72. ***Exploration*** The equation

$$r = \frac{ep}{1 \pm e\sin\theta}$$

is the equation of an ellipse with $e < 1$. What happens to the lengths of both the major axis and the minor axis when the value of e remains fixed and the value of p changes? Use an example to explain your reasoning.

Skills Review

In Exercises 73–78, solve the trigonometric equation.

73. $4\sqrt{3}\tan\theta - 3 = 1$

74. $6\cos x - 2 = 1$

75. $12\sin^2\theta = 9$

76. $9\csc^2 x - 10 = 2$

77. $2\cot x = 5\cos\dfrac{\pi}{2}$

78. $\sqrt{2}\sec\theta = 2\csc\dfrac{\pi}{4}$

In Exercises 79–82, find the exact value of the trigonometric function given that u and v are in Quadrant IV and $\sin u = -\frac{3}{5}$ and $\cos v = 1/\sqrt{2}$.

79. $\cos(u + v)$

80. $\sin(u + v)$

81. $\cos(u - v)$

82. $\sin(u - v)$

In Exercises 83 and 84, find the exact values of $\sin 2u$, $\cos 2u$, and $\tan 2u$ using the double-angle formulas.

83. $\sin u = \dfrac{4}{5}, \ \dfrac{\pi}{2} < u < \pi$

84. $\tan u = -\sqrt{3}, \ \dfrac{3\pi}{2} < u < 2\pi$

In Exercises 85–88, find a formula for a_n for the arithmetic sequence.

85. $a_1 = 0, d = -\frac{1}{4}$

86. $a_1 = 13, d = 3$

87. $a_3 = 27, a_8 = 72$

88. $a_1 = 5, a_4 = 9.5$

In Exercises 89–92, evaluate the expression. Do not use a calculator.

89. ${}_{12}C_9$

90. ${}_{18}C_{16}$

91. ${}_{10}P_3$

92. ${}_{29}P_2$