CHAPTER 1

Metric Spaces

1. Definition and examples

Metric spaces generalize and clarify the notion of distance in the real line. The definitions will provide us with a useful tool for more general applications of the notion of distance:

DEFINITION 1.1. *A metric space is given by a set X and a distance function $d : X \times X \to \mathbb{R}$ such that*

i) *(Positivity) For all $x, y \in X$*

$$0 \le d(x,y) \, .$$

ii) *(Non-degenerated) For all $x, y \in X$*

$$0 \;=\; d(x,y) \quad \Leftrightarrow \quad x \;=\; y \, .$$

iii) *(Symmetry) For all $x, y \in X$*

$$d(x,y) \;=\; d(y,x)$$

iv) *(Triangle inequality) For all $x, y, z \in X$*

$$d(x,y) \;\le\; d(x,z) + d(z,y) \, .$$

Examples:

i) $X = \mathbb{R}$, $d(x,y) = |x-y|$.

ii) $X = \mathbb{R}^2 \;=\; \mathbb{R} \times \mathbb{R}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_1(x,y) \;=\; |x_1 - y_1| + |x_2 - y_2| \, .$$

iii) $X = \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_2(x,y) \;=\; \left(|x_1 - y_1|^2 + |x_2 - y_2|^2\right)^{\frac{1}{2}} \, .$$

iv) Let $X = \{p_1, p_2, p_3\}$ and

$$d(p_1,p_2) \;=\; d(p_2,p_1) \;=\; 1 \, ,$$

$$d(p_1,p_3) \;=\; d(p_3,p_1) \;=\; 2 \, ,$$

$$d(p_2,p_3) \;=\; d(p_3,p_2) \;=\; 3\,.$$

Can you find a triangle (p_1,p_2,p_3) in the plane with these distances?

v) Let $X=\{p_1,p_2,p_3\}$ and

$$\begin{aligned} d(p_1,p_2) &= d(p_2,p_1) \;=\; 1\,,\\ d(p_1,p_3) &= d(p_3,p_1) \;=\; 2\,,\\ d(p_2,p_3) &= d(p_3,p_2) \;=\; 4\,. \end{aligned}$$

Can you find a triangle (p_1,p_2,p_3) in the plane with these distances?

vi) The French railway metric (Chicago suburb metric) on $X=\mathbb{R}^2$ is defined as follows: Let $x_0=(0,0)$ be the origin, then

$$d_{SNCF}(x,y) \;=\; \begin{cases} d_2(x,y) & \text{if there exists a } t\in\mathbb{R} \text{ such that } x_1=ty_1 \\ & \text{and } x_2=ty_2 \\ & \\ d_2(x,x_0)+d_2(x_0,y) & \text{else} \end{cases}\;.$$

Exercise: Show that the railroad metric satisfies the triangle inequality.

It is by no means trivial to show that d_2 satisfies the triangle inequality. In the following we write $0=(0,...,0)$ for the origin in $\mathbb{R}^n$.

CS LEMMA 1.2. *Let $x,y\in\mathbb{R}^n$, then*

$$|\sum_{i=1}^n x_iy_i| \;\le\; \left(\sum_{i=1}^n |x_i|^2\right)^{\frac12}\left(\sum_{i=1}^n |y_i|^2\right)^{\frac12}$$

LEMMA 1.3. *On $\mathbb{R}^n$ the metric*

$$d_2(x,y) \;=\; \left(\sum_{i=1}^n |x_i-y_i|^2\right)^{\frac12}$$

satisfies the triangle inequality.

PROOF. Let $x,y,z\in\mathbb{R}^n$. Then we deduce from Lemma CS 1.2

$$\begin{aligned} d(x,y)^2 &= \sum_{i=1}^n |x_i-y_i|^2 \;=\; \sum_{i=1}^n |(x_i-z_i)-(y_i-z_i)|^2\\ &= \sum_{i=1}^n |(x_i-z_i)|^2 - 2\sum_{i=1}^n (x_i-z_i)(y_i-z_i) + \sum_{i=1}^n |y_i-z_i|^2\\ &\le d(x,z)^2 + 2d(x,y)d(y,z) + d(y,z) \end{aligned}$$

$$= \quad (d(x,z) + d(y,z))^2 \,.$$

Hence,

$$d(x,y) \ \le \ d(x,z) + d(y,z)$$

and the assertion is proved. ■

More examples:

(1) Let n be a prime number. On $\mathbb{Z}$ we define

$$dd_n(x,y) \ = \ n^{-\max\{m\in\mathbb{N}\,:\,n^m \text{ divides x-y}\}} \,.$$

The n-adic metric satisfies a stronger triangle inequality

$$dd_n(x,y) \ \le \ \max\{dd_n(x,z), dd_n(z,y)\} \,.$$

(2) Let $1 \le p < \infty$. Then

$$d_p(x,y) \ = \ \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

defines a metric n $\mathbb{R}^n$.

(3) For $p = \infty$

$$d_\infty(x,y) \ = \ \max_{i=1,..,n} |x_i - y_i|$$

also defines a metric on $\mathbb{R}^n$.

Project 1: Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Show Minkowski's inequality.

Mink (1.1)
$$xy \le \frac{x^p}{p} + \frac{y^q}{q}$$

holds for all $x, y > 0$. **Hint:** the function $f(x) = -\ln x$ is convex on $(0, \infty)$.

Proof of the triangle inequality for d_p. The triangle inequality for $p = 1$ is obvious. We will fist show

mink2 (1.2)
$$|\sum_{i=1}^{n} x_i y_i| \ \le \ \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$. Let $t > 0$. We first observe that

$$|\sum_{i=1}^{n} x_i y_i| = \sum_{i=1}^{n} |tx_i||t^{-1}y_i| \ \le \ \sum_{i=1}^{n} \frac{1}{p}|tx_i|^p + \frac{1}{q}|t^{-1}y_i|^q$$
$$= \frac{t^p}{p} \sum_{i=1}^{n} |x_i|^p + \frac{t^{-q}}{q} \sum_{i=1}^{n} |y_i|^q \,.$$

What is best choice of t? Make

$$t^p \sum_{i=1}^n |x_i|^p \;=\; t^{-q} \sum_{i=1}^n |y_i|^q$$

i.e.

$$t^{p+q} \;=\; \frac{\sum\limits_{i=1}^n |y_i|^q}{\sum\limits_{i=1}^n |x_i|^p} \; .$$

This yields

$$|\sum_{i=1}^n x_i y_i| \le t^p \sum_{i=1}^n |x_i|^p \;=\; \frac{\left(\sum\limits_{i=1}^n |y_i|^q\right)^{\frac{p}{p+q}}}{\left(\sum\limits_{i=1}^n |x_i|^p\right)^{\frac{p}{p+q}}} \sum_{i=1}^n |x_i|^p$$

$$= \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |x_i|^p\right)^{1-\frac{1}{q}}$$

Now, we proof the triangle inequality. Let $x = (x_i)$, (y_i) and $z = (z_i)$ in $\mathbb{R}^d$. Then we apply (mink2 1.2)

$$d_p(x,y)^p = \sum_{i=1}^d |x_i - y_i|^p \;\le\; \sum_{i=1}^d |x_i - y_i|^{p-1}(|x_i - z_i| + |z_i - y_i|)$$

$$\le \sum_{i=1}^d |x_i - y_i|^{p-1}|x_i - z_i| + \sum_{i=1}^d |x_i - y_i|^{p-1}|z_i - y_i|$$

$$\le \left(\sum_{i=1}^d (|x_i - y_i|^{p-1})^q\right)^{\frac{1}{q}} \left(\left(\sum_{i=1}^d |z_i - x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^d |z_i - y_i|^p\right)^{\frac{1}{p}} \right) \; .$$

However, $1 = 1/p + 1/q$ implies $p - 1 = p/q$ and thus $q(p-1) = p$. Hence we get

$$d_p(x,y)^p \;\le\; d_p(x,y)^{p-1}(d_p(x,z) + d_p(z,y)) \; .$$

If $x \neq y$ we may divide and deduce the assertion. ■

2. Excursion: Convex functions

DEFINITION 2.1. *Let I be an interval. A function $f : I \to \mathbb{R}$ is called convex if*

$$f(\lambda x + (1-\lambda)y) \;\le\; \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$, $0 < \lambda < 1$.

LEMMA 2.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous, differnetiable on (a,b) such that f' is increasing. Then f is convex.*

PROOF. Let $x \in [a,b]$. We will show that

$$g(z) \;=\; \frac{f(y+z) - f(y)}{z}$$

is monotone increasing on $(0, b-x)$. Indeed, by the fundamental theorem and change of variables we deduce for $z_1 < z_2$ and $\lambda \;=\; \frac{z_1}{z_2}$ $(s = \lambda t$, $ds = \lambda dt$

$$g(z_1) = \int_0^{z_1} f'(s)\frac{ds}{z_1} \;=\; \int_0^{z_2} f'(\lambda t)\frac{\lambda dt}{z_1} \;=\; \int_0^{z_2} f'(\lambda t)\frac{dt}{z_2}$$
$$\le \int_0^{z_2} f'(t)\frac{dt}{z_2} \;=\; g(z_2)\,.$$

Now, we fix $y < x$ and $u = \lambda x + (1-\lambda)y = y + \lambda(x-y)$, $z_1 = \lambda(x-y)$, $z_2 = x-y$. Then, we get

$$\frac{f(y+z) - f(y)}{\lambda(x-y)} \;\le\; \frac{f(x) - f(y)}{(x-y)}\,.$$

This implies

$$f(\lambda x + (1-\lambda)y) \le f(y) + \lambda(f(x) - f(y)) \;=\; \lambda f(x) + (1-\lambda)f(y)\,.$$ ■

PROOF OF Mink 1.1. Let $x, y > 0$. Since $-\ln x$ is convex we have

$$-\ln(\frac{1}{p}x^p + \frac{1}{q}y^q) \;\le\; \frac{1}{p}(-\ln x^p) + \frac{1}{q}(-\ln y^q)\,.$$

This shows by the monotonicity of exp that

$$\frac{1}{p}x^p + \frac{1}{q}y^q \;\ge\; e^{\ln x + \ln y} \;=\; xy\,.$$

Minkowski's inequality is proved. ■

3. Continuous functions between metric spaces

Continuous functions 'preserve' properties of metric spaces and allow to describe deformation of one metric space into another. There are three different (but equivalent) ways of defining continuity, the ε-δ-criterion, the sequence criterion and the topological criterion. Each of them is interesting in its own right.

Definition 3.1. *Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is called continuous if for every $x \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that*

edelt
$$d(x, y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon \, . \tag{3.1}$$

Let us use the notation

$$B(x, \delta) \;=\; \{y \,:\, d(x, y) < \delta\} \, .$$

For a subset $A \subset X$, we also use the notation

$$f(A) \;=\; \{f(x) \,:\, x \in A\} \, .$$

Similarly, for $B \subset Y$

$$f^{-1}(B) \;=\; \{x \in X \,:\, f(x) \in B\} \, .$$

Then (edelt 3.1) means

$$f(B(x, \delta)) \;\subset B(f(x), \varepsilon) \, .$$

Or in a very non-formal way

$$\text{f maps small balls into small balls} \quad .$$

Our aim is to prove a criterion for continuity in terms of so called open sets. This criterion illustrates simultaneously the role of open sets and its interaction with continuity and has a genuinely geometric flavor.

Definition 3.2. *A subset O of a metric space is called open if*

$$\forall x \in O \;: \exists \delta > 0 \;: B(x, \delta) \subset O \, .$$

Examples:

$$O \;=\; (-1, 1) \, , \; O \;=\; \mathbb{R} \, , \; O \;=\; (-1, 1) \times (-2, 2)$$

are open in $\mathbb{R}$, $(\mathbb{R}^2, d_2)$ respectively.

Remark 3.3. *The sets $B(x, \varepsilon)$, $x \in X$, $\varepsilon > 0$ are open.*

Proposition 3.4. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. f is contiuous iff $f^{-1}(O)$ is open for all open subsets $O \subset Y$.*

PROOF. $\Rightarrow$: We assume that f is continuous and O is open. Let $x \in f^{-1}(O)$, i.e. $f(x) \in O$. Since O is open, there exists an $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset O$. By continuity, there exists a $\delta > 0$ such that

$$f(B(x,\delta)) \subset B(f(x),\varepsilon) \subset O\ .$$

Therefore

$$B(x,\delta) \subset f^{-1}(O)\ .$$

Since $x \in f^{-1}(O)$ was arbitrary, we deduce that $f^{-1}(O)$ is open.
$\Leftarrow$: Let $x \in X$ and $\varepsilon > 0$. Let us show that

$$B(f(x),\varepsilon)$$

is a on open subset of (Y, d'). Indeed, let $y \in B(f(x), \varepsilon)$ define $\varepsilon' = \varepsilon - d'(y, f(x))$. Let $z \in Y$ such that

$$d(z,y) < \varepsilon'$$

then

$$d(f(x),z) \ \leq\ d(f(x),y) + d(y,z) \ <\ d(f(x),y) + \varepsilon - d'(y,f(x)) \ =\ \varepsilon\ .$$

Thus

$$B(y, \varepsilon - d'(f(x),y)) \ \subset B(f(x),\varepsilon)\ .$$

By the assumption, we see that $f^{-1}(B(f(x), \varepsilon))$ is an open set. Since $x \in f^{-1}(B(f(x), \varepsilon))$, we can find a $\delta > 0$ such that

$$B(x,\delta) \subset f^{-1}(B(f(x),\varepsilon))\ .$$

Hence, for all $\tilde{x}$ with $d(x, \tilde{x}) < \delta$, we have

$$d'(f(x),f(\tilde{x})) < \varepsilon\ .$$

The assertion is proved. ∎

Examples:

(1) Let (X, d) be a metric space and $x_0 \in X$ be a point , then $f(x) = d(x, x_0)$ is continuous. Indeed, the triangle inequality implies

$$d(d(x,x_0), d(d(y,0)) \ =\ |d(x,x_0) - d(y,x_0)| \ \leq\ d(x,y)$$

This easily implies the assertion.

(2) On $\mathbb{R}^n$ with the standard euclidean metric $d = d_2$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = d(x, 0)x$ is continuous.

(3) (Exercise) The function $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\cos(x_1), \sin(x_2), \cos(x_1))$ is continuous.

Definition 3.5. *Let (X, d), (Y, d') be a metric space. The space $C(X, Y)$ is the set of all continuous functions from X to Y. Let $x_0 \in X$ be a point. Then*

$$C_b(X,Y) \;=\; \{f : X \to Y \;:\; f \text{ is continuous and } \sup_{x\in X} d'(f(x), f(x_0)) < \infty\}$$

is the subset of bounded continuous functions.

Proposition 3.6. *Let (X, d), (Y, d') be metric spaces and $x_0 \in X$. Then $C_b(X, Y)$ equipped with*

$$d(f,g) \;=\; \sup_{x\in X} d'(f(x), g(x))$$

is a metric space.

Problem: Show that d is not well-defined on $C(\mathbb{R}, \mathbb{R})$.

Proof: $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all $x \in X$. This means $f = g$. Let us show that d is well-defined. Indeed, if $f, g \in C_b(X, Y)$. Then

$$\begin{aligned}\sup_x d'(f(x), g(x)) &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + d'(g(x_0), g(x)) \\ &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + \sup_x d(g(x_0), g(x))\end{aligned}$$

is finite. Let h be a third function and $x \in X$. Than

$$d'(f(x), g(x)) \;\le\; d'(f(x), h(x)) + d(h(x), g(x)) \;\le\; d(f, h) + d(h, g)\,.$$

Taking the supremum yields the assertion. ∎

alg Proposition 3.7. *Let (X, d) be a metric space. Then $C(X, \mathbb{R})$ is closed under (pointwise-) sums, products and multiplication with real numbers. ($C(X, \mathbb{R})$ is an algebra over $\mathbb{R}$).*

Remark 3.8. *Let $X = \mathbb{N}$ and $d(x, y) = 1$ of $x \neq y$ and $d(x, y) = 0$ for $x = y$. (This is called the discrete metric). Then $C(X, \mathbb{R})$ is an infinite dimensional vector space.*

Proof of alg 3.7. Let $f, g \in C(X, \mathbb{R})$ be continuous and $x \in X$. Consider $x' \in X$. Then

$$\begin{aligned} fg(x) - fg(y) \;&=\; f(x)g(x) - f(y)g(y) \;=\; (f(x) - f(y))g(x) + f(y)(g(x) - g(y)) \\ &= (f(x) - f(y))g(x) + f(x)(g(x) - g(y)) + (f(y) - f(x))(g(x) - g(y))\,. \end{aligned}$$

Let $\varepsilon > 0$ and $\tilde{\varepsilon} = \min\{\varepsilon, 1\}$. We may choose $\delta_1 > 0$ such that

$$d(f(x), f(y))(1 + |g(x)|) < \frac{\tilde{\varepsilon}}{3}$$

holds for all $d(x,y) < \delta_1$. Similarly, we may choose $\delta_2 > 0$ such that

$$d(g(x), g(y))(1 + |f(x)|) < \frac{\tilde{\varepsilon}}{3} .$$

Let $\delta = \min(\delta_1, \delta_2)$ and $d(x,y) < \delta$. Then we deduce that

$$d(fg(x), fg(y)) = |fg(x) - fg(y)| < \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}^2}{9} < \tilde{\varepsilon} \le \varepsilon .$$

Thus fg is again continuous. The other assertions are easier. ■

Corollary 3.9. *The polynomials on $\mathbb{R}$ are continuous.*

Lemma 3.10. *Let $1 \le p \le \infty$ and $x, y \in \mathbb{R}^n$, then*

$$\frac{1}{n^{\frac{1}{p}}} d_p(x,y) \le d_\infty(x,y) \le d_p(x,y) .$$

Proof. The last inequality is obvious. For the first one, we consider $x, y \in \mathbb{R}^n$ and $1 \le p < \infty$, then by estimating every element in the sum against the maximum

$$d_p(x,y)^p = \sum_{i=1}^{n} |x_i - y_i|^p \le n \max\{|x_i - y_i|^p\} .$$

Taking the p-th root, we deduce the assertion. ■

Corollary 3.11. *Let $1 \le p, q \le \infty$, then the identity map $id : (\mathbb{R}^n, d_p) \to (\mathbb{R}^n, d_q)$ is continuous.*

Proof. We have for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$

$$B_{d_p}(x, \frac{\varepsilon}{n}) \subset B_{d_q}(x, \varepsilon) .$$

This easily implies the assertion. ■

Corollary 3.12. *The metrics d_p define the same open sets on $\mathbb{R}^n$.*

Definition 3.13. *Let (X,d) be a metric space. We say that a sequence (x_n) converges to x if for all $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ we have*

$$d(x_n, x_0) < \varepsilon .$$

In this case we write

$$\lim_n x_n = x$$

or more explicitly

$$d - \lim_n x_n = x .$$

A sequence (x_n) is convergent, if there exists $x \in X$ with $\lim_n x_n = x$.

Examples: $d_2 - \lim_n \frac{1}{n} = 0$, $dd_3 - \lim_n 3^n = 0$. (What axioms of the natural numbers are involved?).

PROPOSITION 3.14. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. Then f is continuous if for every convergent sequence (x_n) in X*

$$\lim_n f(x_n) = f(\lim_n x_n) .$$

Proof: $\Rightarrow$: Let $x = \lim_n x_n$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that

$$d(y, x) < \delta \Rightarrow d'(f(y), f(x)) < \varepsilon .$$

Let $n_0 \in \mathbb{N}$ be such that

$$d(x_n, x) < \delta$$

for all $n > n_0$, then

$$d'(f(x_n), f(x)) < \varepsilon$$

for all $n > n_0$. Hence

$$\lim_n f(x_n) = f(x) .$$

$\Leftarrow$ Let $x \in X$ and assume in the contrary that

$$\exists \varepsilon > 0 \, \forall \delta > 0 \exists y \ : d(y, x) < \delta \text{ and } d'(f(x), f(y)) \geq \varepsilon .$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence (x_k) such that

$$d(x_k, x) < \frac{1}{k} \quad \text{and} \quad d'(f(x_k), f(x)) \geq \varepsilon' .$$

and thus

$$\lim_k x_k = x .$$

By assumption, we have

$$\lim_k f(x_k) = f(x) .$$

Hence, there exists a k_0 such that for all $k > k_0$

$$d(f(x_k), f(x)) < \varepsilon .$$

a contradiction. ■

4. Complete metric spaces and completion

Complete metric space are crucial in understanding existence of solutions to many equations. Complete spaces are also important in understanding spaces of integrable functions. We will review basic properties here and show the existence of a completion.

We will say that a sequence in a metric space is a Cauchy sequence of for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$.

DEFINITION 4.1. *A metric space (X, d) is called complete, if every Cauchy sequence converges.*

model PROPOSITION 4.2. *The space $(\mathbb{R}^2, d_1)$ is complete.*

Proof: Let x_n be a Cauchy sequence in $(\mathbb{R}^2, d_1)$. Then $x_n = (x_n(1), x_n(2))$ is a sequence of pairs.

Claim: The sequences $(x_n(1))_{n\in\mathbb{N}}$ and $(x_n(2))_{n\in\mathbb{N}}$ are Cauchy sequences.

Indeed, let $\varepsilon > 0$, then there exists an n_0 such that

$$d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$. In particular, we have

$$|x_n(1) - x_m(1)| \;\le\; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \;\le\; d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$ and

$$|x_n(2) - x_m(2)| \;\le\; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \;\le\; d_1(x_n, x_m) < \varepsilon \,.$$

Therefore, $(x_n(1))$ and $(x_n(2))$ are Cauchy.

Since $\mathbb{R}$ is complete, we can find $x(1)$ and $x(2)$ such that

$$\lim_n x_n(1) \;=\; x(1) \quad \text{and} \quad \lim_n x_n(2) \;=\; x(2)\,.$$

Claim: $\lim_n x_n \;=\; (x(1), x(2))$.

Indeed, Let $\varepsilon > 0$ and choose n_1 such that

$$|x_n(1) - x(1)| < \frac{\varepsilon}{2}$$

for all $n > n_1$. Choose n_2 such that

$$|x_n(2) - x(2)| < \frac{\varepsilon}{2}$$

for all $n > n_2$. Set $n_0 = \max\{n_1, n_2\}$, then for every $n > n_0$, we have

$$d_1(x_n, (x(1), x(2)) = |x_n(1) - x(1)| + |x_n(2) - x(2)| < \varepsilon$$

Thus

$$\lim_n x_n = x$$

and the assertion is proved. ∎

Examples:

(1) Let $X = \mathbb{R} \setminus \{0\}$ and $d(x, y) = |x - y|$, them (X, d) is not complete. The sequences $(\frac{1}{n})$ is Cauchy and does not converge.

(2) Let p be a prime number. On the set of integers, we define

$$dd_p(z, w) = p^{-n} ,$$

where $n = \max\{n : p^n \text{ divides } (z-w)\}$. This satisfies the triangle inequality. The sequence (x_n) given by $x_n = p + p^2 + \cdots + p^n$ is a non convergent Cauchy sequence.

rncomp THEOREM 4.3. *Let $n \in \mathbb{N}$. The space $(\mathbb{R}^n, d_2)$ is a complete metric space.*

PROOF. Similar as in Proposition model 4.2 using the following Lemma . ∎

LEMMA 4.4. *Let $x, y \in \mathbb{R}^n$, then*

$$d_2(x, y) \le \sum_{i=1}^{n} |x_i - y_i| .$$

PROOF. We proof this by induction on $n \in \mathbb{N}$. The case $n = 1$ is obvious. Assume the assertion is true for n and let $x, y \in \mathbb{R}^{n+1}$. We define the element $z = (x_1, ..., x_n, y_{n+1})$, then we deduce from the triangle inequality

$$\begin{aligned} d_2(x, y) &\le d_2(x, z) + d_2(z, y) \\ &= \left(\sum_{i=1}^{n+1} |x_i - z_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n+1} |z_i - y_i|^2\right)^{\frac{1}{2}} \\ &= |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} . \end{aligned}$$

To apply the induction hypothesis, we define $\tilde{x} = (x_1, ..., x_n)$ and $\tilde{y} = (y_1, .., y_n)$. Then the induction hypothesis yields

$$\left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} = d_2(\tilde{x}, \tilde{y}) \le \sum_{i=1}^{n} |x_i - y_i| .$$

Hence,

$$\begin{aligned} d_2(x,y) &\leq |x_{n+1}-y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} \\ &\leq |x_n - y_n| + \sum_{i=1}^{n} |x_i - y_i| \\ &= \sum_{i=1}^{n+1} |x_i - y_i| \,. \end{aligned}$$

The assertion is proved. ■

Definition 4.5. *A subset $C \subset X$ is called closed if $X \setminus C$ is open.*

complete1 Proposition 4.6. *Let C be closed subset of a complete metric space (X,d), then $(C, d|_{C\times C})$ is complete.*

Proof. Let $(x_n) \subset C$ be Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$x = \lim_n x_n \,.$$

We have to show $x \in C$. Assume $x \notin C$. Then there exists a $\delta > 0$ such that $B(x,\delta) \subset X \setminus C$. By definition of the limit there exists n_0 such that $d(x_n, x) < \delta$ for all $n > n_0$. Set $n = n_0 + 1$. Then $d(x_n, x) < \delta$ implies $x_n \in X \setminus C$ and $x_n \in C$ by definition. This contradiction finished the proof. ■

comp1 Theorem 4.7. *Let (Y,d') be complete metric space. Let $h \in C(X,Y)$ and*

$$C_h(X,Y) = \{f \in C(X,Y) \,:\, \sup_{x\in X} d'(f(x),h(x)) < \infty\}$$

Then $C_g(X,Y)$ is complete with respect to

$$d(f,g) = \sup_{x\in X} d'(f(x),g(x)) \,.$$

Proof. Let $(f_n) \subset C_h(X,Y)$ be Cauchy sequence. This means that for every $\varepsilon > 0$ there exists an n_0 such that

eqq (4.1) $$\sup_{x\in X} d'(f_n(x), f_m(x)) < \frac{\varepsilon}{2} \,.$$

In particular, for fixed $x \in X$, $f_n(x)$ is Cauchy. Therefore $f(x) := \lim_m f_m(x)$ is a well-defined element in Y. We fix $n > n_0$ and consider $m \geq n_0$ such that

$$d'(f_m(x), f(x)) \leq \frac{\varepsilon}{3} \,.$$

This implies

$$d'(f_n(x), f(x)) \;\leq\; d'(f_n(x), f_m(x)) + d'(f_m(x), f(x)) \;\leq\; \frac{5}{6}\varepsilon$$

for all $x \in X$. In particular,

eqq2

$$\sup_{n \geq n_0} \sup_{x \in X} d'(f_n(x), f(x)) \;\leq\; \frac{5}{6}\varepsilon \,. \tag{4.2}$$

Let us show that f is continuous. Let $z \in X$ and $\varepsilon > 0$. Choose n_0 according to (4.1). Choose $n = n_0 + 1$. Let $\delta > 0$ such that $d(x, y) < \delta$ implies $d'(f_n(x), f_n(y)) < \varepsilon$. Then, we have

$$d'(f(x), f(y)) \;\leq\; d'(f(x), f_n(x)) + d'(f_n(x), f_n(y)) + d'(f_n(y), f(y)) < 3\varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, we see that f is continuous. Moreover, (4.2) implies that f_n converges to f. Finally, (4.2) for $\varepsilon = 1$ implies that

$$\sup_x d(f(x), h(x)) \;\leq\; \sup_x d(f(x), f_n(x)) + \sup_x d(f_n(x), h(x)) < \infty$$

implies that $f \in C_h(X, Y)$. ■

Definition 4.8. *Let (X, d) be a metric space and $C \subset X$. $O \subset X$ is called sense if for ever $x \in C$ and $\varepsilon > 0$ $B(x, \varepsilon) \cap O \neq \emptyset$.*

Definition 4.9. *Let $O \subset X$ be a subset. Then*

$$\bar{O} \;=\; \cap_{O \subset C, C\, closed} C$$

is called the closure.

dens

Lemma 4.10. *O is dense in $\bar{O}$ and $\bar{O}$ is closed.*

Proof. Let $x \in \bar{O}$. Assume $B(x, \varepsilon) \cap O \;=\; \emptyset$. Then $C = X \setminus B(x, \varepsilon)$ contains O. Thus

$$\bar{O} \subset C \,.$$

This implies that $x \notin \bar{O}$, a contradiction. Now, we show that $\bar{O}$ is closed. Indeed, let $y \notin \bar{O}$. Then there has to be a closed set C such that $O \subset C$ but $y \notin C$. This means $y \in X \setminus C$ which is open. Hence there exists $\delta > 0$ such that

$$B(y, \delta) \subset X \setminus C$$

By definition every element $z \in B(y, \delta)$ does not belong to $\bar{O}$. This means $B(y, \delta) \subset X \setminus \bar{O}$. ■

comp2 THEOREM 4.11. *Let (X, d) be a non-empty metric space. For every $x \in X$ we define*

$$f_x(y) \ = \ d(x, y) \ .$$

Let $x_0 \in X$. The map $f : X \to C_{f_{x_0}}(X, \mathbb{R})$ satisfies the following properties.

i) $d(f(x), d(f(y)) \ = \ d(x, y)$,

(1) *The closure $C = \overline{f(X)}$ is complete,*

(2) *$f(X)$ is dense in the closure $C = \overline{f(X)}$.*

PROOF. Let $x, y \in X$ and $z \in X$. Then the 'converse triangle' ineqquality implies

$$|f_x(z) - f_y(z)| \ = \ |d(x, z) - d(y, z)| \ \leq \ d(x, y) \ .$$

Moreover,

$$|f_x(z) - f_x(y)| \ = \ |d(x, z) - d(x, y)| \ \leq \ d(z, y) \ .$$

Therefore $f_x \in C_{f_{x_0}}(X, \mathbb{R})$ for every $x \in X$ and

$$d(f_x, f_y) \ \leq \ d(x, y) \ .$$

However,

$$d(f_x, f_y) \ \geq \ |f_x(x) - f_y(x)| \ = \ |0 - d(y, x)| \ = \ d(y, x) \ .$$

This shows i). According to Proposition complete1 4.6 and Theorem comp1 4.7, we see that C is complete. According to Lemma dens 4.10, we deduce that $f(X)$ is dense in C. ■

Project: On $C([0, 1])$ we define

$$d_1(f, g) \ = \ \int |f(s) - g(s)| ds \ .$$

Show that $(C([0, 1]), d_1)$ is not complete.

Project: In the literature you can find another description of the completion of a metric space. Find it and describe it.

5. Unique extension of densely defined uniformly continuous functions

In this section we will show that the completion C constructed in Theorem comp2 4.11 is unique (in some sense). This is based on a simple observation-the unique extension. This principle is very often used in analysis.

DEFINITION 5.1. *Let (X, d), (Y, d') be metric spaces. A function $f : X \to Y$ is called uniform continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$d(x, y) < \delta \quad \Rightarrow \quad d'(f(x), f(y)) < \varepsilon \ .$$

u-ext PROPOSITION 5.2. *Let $O \subset C$ be a dense set and $f : O \to Y$ be uniformly continuous function with values in a complete metric space. Then there exists a unique continuous function $\tilde{f} : O \to Y$ such that $\tilde{f}(x) = f(x)$ for all $x \in O$.*

PROOF. Let $x \in X$. Since $B(x, \frac{1}{n}) \cap O$ is not empty, we may find $(x_n) \subset O$ such that $\lim_n x_n = x$. We try to define

$$\tilde{f}(x) \;=\; \lim_n f(x_n) \,.$$

Let us show that this is well-defined. So we consider another Cauchy sequence (x'_n) such that $\lim_n x'_n = x$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$d'(f(x', y)) < \varepsilon$$

holds for $d(x', y) < \delta$. We may find n_0 such that

$$d(x_n, x) < \frac{\delta}{2}$$

and

$$d(x'_n, x) < \frac{\delta}{2}$$

holds for all $n, n' > n_0$. Thus

$$d'(f(x'_n), f(x_n)) < \varepsilon \,.$$

This argument also shows that $(f(x_n))$ is Cauchy and hence $\tilde{f}(x)$ is well-defined. If $x \in O$, we may choose for (x_n) the constant sequence $x_n = x$ and hence $\tilde{f}(x) = f(x)$. Now, we want to show that $\tilde{f}$ is uniformly continuous. Indeed, let $\varepsilon > 0$, then there exists $\delta > 0$ such that $d(x', y') < \delta$ implies

$$d(f(x'), f(y')) < \frac{\varepsilon}{2} \,.$$

Given $x, y \in C$ with $d(x, y) < \delta$, we may find (x_n) converging to x and (y_n) converging to y such that

$$d(x_n, x) < \frac{\delta - d(x, y)}{2} \,.$$

Thus for all $n \in \mathbb{N}$ we have

$$d(x_n, y_n) \;\le\; d(x, y) + d(x_n, x) + d(y_n, y) < \delta \,.$$

This implies

$$d(f(x), f(y)) \;=\; \lim_n d(f(x_n), f(y_n)) < \frac{\varepsilon}{2} \,.$$

This shows that $\tilde{f}$ is uniformly continuous. If g is another continuous function such that $g(x) = f(x)$ holds for elements $x \in O$, then we may choose a Cauchy sequence (x_n) converging to x and get

$$g(x) = \lim_n g(x_n) \;=\; \lim_n f(x_n) \;=\; f(x)\,. \qquad \blacksquare$$

Example If $f : (0,1] \to \mathbb{R}$ is uniformly continuous, then f is bounded (why). $f(x) = 1/x$ is not uniformly continuous.

THEOREM 5.3. *The completion of a metric space is unique. More precisely, let C be the set constructed in Theorem* comp2 *4.11. Let C' be a complete metric space and $\iota' : X \to C'$ be uniformly continuous with uniformly continuous inverse ${\iota'}^{-1} : \iota(X) \to X$ such that $\iota'(X)$ is dense. Then there is a bijective, bicontinuous map $u : C \to C'$ such that $u(\iota(x)) = \iota'(x)$.*

PROOF. The map $\iota'\iota^{-1} : \iota(X) \to C'$ is uniformly continuous and hence admits a unique continuous extension $u : C \to C'$. Also $\iota{\iota'}^{-1} : \iota'(X) \to C$ admits a unique extension $v : C' \to C$. Note that $vu : C \to C$ is an extension of the map $vu(\iota(x)) = \iota(x)$. Thus there is only one extension, namely the identity. This show $vu = id$. Similarly $uv = id$. Thus $v = u^{-1}$ and u is bijective and bi-continuous. $\blacksquare$

Project: Find the completion of $(\mathbb{Z}, d_3)$.

6. A famous example

In this section we want to identify the completion of $C([0,1])$ with respect to

$$d_1(f,g) \;=\; \int_0^1 |f(t) - g(t)|dt\,.$$

We will also use the function

$$I(f) \;=\; \int_0^1 f(t)dt$$

defined by the Riemann integral.

LEMMA 6.1. *I is uniformly continuous.*

PROOF. It suffices to show that

$$|I(f)| \;\le\; \int |f(t)|dt\,.$$

(This implies that I is 1-Lipschitz, i.e.

$$|I(f) - I(g)| \;\le\; d_1(f,g)\ .)$$

We define $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then we have

$$I(f) \;=\; I(f^+) - I(f^-) \;\le\; I(f^+) + I(f^-) \;=\; I(|f|) \;=\; \int |f(t)|dt\ .$$

Similarly, we may show that

$$I(f) \;=\; I(f^+) - I(f^-) \;\ge\; -I(f^+) - I(f^-) \;=\; -I(|f|) \;=\; -\int |f(t)|dt\ .$$

The assertion follows. ■

The characteristic function is given by

$$1_A(x) \;=\; \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}\ .$$

Lemma 6.2. *$I(1_{[a,b]}) \;=\; b - a$.*

Proof. We only consider $[a,b] = [0,b]$. For $2/n \le b$ we define

$$f_n(t) \;=\; \begin{cases} nt & \text{if } t \le \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le t \;=\; b - \frac{1}{n} \\ n(b-t) & \end{cases}\ .$$

Then we deduce that for $m \ge n$ we have

$$d_1(f_n, f_m) \;=\; 2\int_0^{\frac{1}{m}} mt\,dt + \frac{1}{n} - \frac{1}{m} - \int_0^{\frac{1}{n}} ntdt \;=\; 2(\frac{1}{2m} + \frac{1}{n} - \frac{1}{m} - \frac{1}{2n}) \;=\; \frac{1}{n} - \frac{1}{m}\ .$$

Thus (f_n) is Cauchy and

$$\lim_n \int f_n(t)dt \;=\; \lim_n b - \frac{1}{n} \;=\; b\ .$$

For general a we simply shift. ■

In the following we denote the length of an interval by

$$|[a,b]| \;=\; b - a\ .$$

chy **LEMMA 6.3.** *Let f be a continuous positive function on $[0,1]$, $\lambda > 0$ and $\varepsilon > 0$. Then there exists intervals $J_1, ..., J_m$ such that*

$$\{t \,:\, f(t) > \lambda\} \subset J_1 \cup \cdots J_k$$

and

$$\sum_k |J_k| \;\le\; \frac{1}{\lambda - 2\varepsilon} \int f(t)dt\,.$$

PROOF. Let $\varepsilon < \frac{\lambda}{2}$. Since f is uniformly continuous, we may find $n \in \mathbb{N}$ such that $|x-y| \le 1/n$ implies

$$|f(x) - f(y)| < \varepsilon\,.$$

We define $x_k \;=\; k/n$. Let $S = \{k : f(x_k) > \lambda - \varepsilon\}$. Let $t \in [0,1]$ such that $f(t) > \lambda$. We consider $k = \lfloor tn \rfloor$. Then $x_k - t \le \frac{1}{n}$ and hence

$$f(x_k) > f(t) - \varepsilon > \lambda - \varepsilon\,.$$

Therefore

$$\{t : f(t) > \lambda\} \subset \bigcup_{k\in S} [x_k, x_{k+1}]\,.$$

However, $f(x_k) > \lambda - \varepsilon$ implies $f(t) > \lambda - 2\varepsilon$ for all $t \in [x_k, x_{k+1}]$. By the definition of the lower sums we deduce

$$\int_0^1 f(t)dt \;\ge\; \sum_{k\in S} (\lambda - 2\varepsilon)|J_k|\,.$$

Since $\lambda - 2\varepsilon > 0$, we deduce the assertion. ■

DEFINITION 6.4. i) *A subset $A \subset [0,1]$ is said to have measure 0 if for every $\varepsilon > 0$ there exists a sequence (J_k) of intervals such that*

$$A \subset \bigcup_k J_k \quad \text{and} \quad \sum_k |J_k| < \varepsilon\,.$$

ii) *A sequence (f_n) converges almost everywhere (a.e.) to a function f if there exists a set A of measure 0 such that*

$$\lim_n f_n(t) \;=\; f(t)$$

for all $t \in A^c = [0,1] \setminus A$.

limit1 **PROPOSITION 6.5.** *Let (f_n) be a d_1-Cauchy sequence. Then there exists a subsequence (n_k) and a function f such that f_{n_k} converges to f a.e.*

PROOF. We may choose (n_k) such that

$$d_1(f_{n_k}, f_{n_{k+1}}) \ \leq \ 6^{-k} \ .$$

Let us denote $g_k = f_{n_k}$. We apply Lemma 6.3 to $\lambda = 2^{-k}$ and $\varepsilon = \frac{2^{-k}-3^{-k}}{2}$ and find intervals $J_1^k, ..., J_{m(k)}^k$ such that

$$\sum_l |J_l^k| \ \leq \ \frac{6^{-k}}{2^{-k} - 2\varepsilon} \ = \ 2^{-k}$$

and

$$\{t \in [0,1] : |g_k(t) - g_{k+1}(t)| > 2^{-k}\} \subset \bigcup_l J_l^k \ .$$

We define

$$A_k \ = \ \bigcup_{n \geq k} \bigcup_l J_l^n$$

and

$$A \ = \ \bigcap_k A_k \ .$$

Then, we see that $A \subset A_k$ and

$$\sum_{n \geq k} \sum_l |J_l^n| \ \leq \ \sum_{n \geq k} 2^{-n} \ = \ 2^{1-k} \ .$$

Thus shows that A has measure 0. For $t \notin A$, we may find k such that for all $n \geq k$ we have $t \notin \bigcup_l J_l^n$. This implies

$$|g_n(t) - g_{n+1}(t)| \ \leq \ 2^{-n}$$

for all $n \geq k$. In particular, $(g_k(t))$ is Cauchy for all $t \in A^c$. We may define

$$f(t) \ = \ \begin{cases} \lim_k g_k(t) & \text{if } t \notin A \\ 0 & \text{else} \end{cases} \ .$$

Then (g_k) converges to f almost everywhere. ■

This leads us to define the set of possible limits.

$$\mathcal{L} \ = \ \{f : [0,1] \to \mathbb{R} \ : \ \exists (f_n) \subset C[0,1], f_n \text{ converges to } f \text{ a.e.}\}$$

on $\mathcal{L}$ we define the equivalence relation $f \sim g$ if there exists a set A of measure 0 such that $f 1_{A^c} = g 1_{A^c}$.

Exercise: Show that $\sim$ is an equivalence relation.

We define

$$L \ = \ \mathcal{L}/\sim \ .$$

For a function $f \in \mathcal{L}$ we define the equivalence class $[f] = \{g : g \sim f\}$. In the following we denote by X the completion of $C[0,1]$ with respect to the d_1-metric. Our main theorem is the following.

inj THEOREM 6.6. *There is an injective map $\iota : X \to L$ such that*

$$\iota(x) \;=\; [f]$$

holds whenever (f_n) is a Cauchy sequence converging to x (with respect to d_1) and converging to f. a.e. Moreover, I can be extended to $\iota(X)$.

Problem: Give a description of $\iota(X_1)$. This is done in the real analysis course (441=540).

We need some more preparation.

uniqueext LEMMA 6.7. *Let $A = \bigcup_k J_k$ the union of intervals.*

i) *Let $f \in C[0,1]$, then $f1_A \in X$, $f1_{A^c} \in X$ and*

$$d_1(f1_A, g1_A) \;\le\; d_1(f,g)$$

and

$$d_1(f1_{A^c}, g1_{A^c}) \;\le\; d_1(f,g)\,.$$

ii) *There is are continuous maps $m_A : X \to X$, $m_{A^c} : X \to X$ such that $m_A(f) = f1_A$ and $m_{A^c}(f) = f1_{A^c}$ for $f \in C[0,1]$.*

iii) *There is a Lipschitz map $add : X \times X \to X$ such that $add(f,g) = f+g$.*

iv) *$add(m_A(x), m_{A^c}(x)) = x$ for all $x \in X$.*

v) $d_1(f1_{A^c}, 0) \;\le\; \sup_{t\in A^c} |f(t)|$.

PROOF. We will start with i) for $A = [a,b]$. We use the functions Let f_n defined for $[0, b-a]$ and define $g_n(t) \;=\; f_n(t-a)$. Then we see that for every $f \in C[0,1]$ we have

$$d_1(ff_n, ff_m) = \int_a^b |f(t)(f_n(t) - f_m(t))|dt \;\le\; \sup_t |f(t)| d(f_n, f_m)$$

$$\le \sup_t |f(t)| d_1(f_n, f_m) \;\le\; \sup_t |f(t)|\, (\frac{1}{n} - \frac{1}{m})\,.$$

Thus (ff_n) is Cauchy. We denote the limit by $f1_{[a,b]}$. (Moreover, ff_n converges pointwise to $f1_{[a,b]}$.) Now, we observe that $|f_n(t)| \le 1$ and hence

$$d_1(f1_{[a,b]}, g1_{[a,b]}) = \lim_n d_1(ff_n, gf_n) \;=\; \lim_n \int_0^1 |f_n(t)(f(t) - g(t))|dt$$

$$\leq \int_0^1 |f(t) - g(t)|dt \;=\; d_1(f,g)\;.$$

By Proposition 5.2 we find a map $u_A : X \to X$ such that $u_A(f) = f1_A$ and

$$d_1(u_A(x), u_A(y)) \leq d_1(x,y)\;.$$

Now, we will prove iii). The metric on $X \times X$ is given by

$$d((x,y),(x',y')) = d_1(x,x') + d_1(y,y')\;.$$

Now, we consider $add : C[0,1] \times C[0,1] \to C[0,1]$ and want to show that add is uniformly continuous. Indeed, elementary properties of the integral imply

$$\begin{aligned} d_1(add(f,g), add(f',g')) &= \int_0^1 |(f+g) - (f'+g')|dt \\ &\leq \int_0^1 |f-f'|dt + \int_0^1 |g-g'|dt \;=\; d((f,g),(f',g')\;. \end{aligned}$$

Thus Proposition 5.2 implies the assertion iii). In the nest step we prove i) for $A = J_1 \cup \cdots \cup J_n$. The key observation here is that we can find new intervals $J_1', ..., J_m'$ such that the J_l' only overlap in one point and

$$A \;=\; \bigcup_l J_l'\;.$$

Therefore, we may define

$$u_A(x) \;=\; \sum_{l=1}^m 1_{J_l'}x \;=\; add(1_{J_1'}x, add(1_{J_2'}x, \cdots add(1_{J_{m-1}'}x, 1_{J_m'}x)\cdots)\;.$$

Being a composition of continuous function that is continuous. Moreover, for every l we may consider the sequence of function f_n^l constructed for the interval J_l'. The function

$$f_n(t) \;=\; \sum_{l=1}^m f_n^l(t)$$

is positive, continuous and vanishes in the overlapping endpoints. Hence $0 \leq f_n(t) \leq 1$ and the argument from above shows

$$d_1(f_nf, f_ng) \leq d_1(f,g)\;.$$

This yields

$$d_1(f1_A, g1_A) \;=\; \lim_f d_1(ff_n, gf_n) \;\leq\; d_1(f,g)\;.$$

Then we define

$$f1_{A^c} \;=\; f - f1_A \;=\; \lim_n add(f, -ff_n) \;=\; \lim_n f(1-f_n)\,.$$

Since $0 \le 1 - f_n \le 1$ we also prove as above that

$$d_1(f1_{A^c}, g1_{A^c}) \;\le\; d_1(f,g)\,.$$

Therefore i) is proved for A being a finite union of intervals. Let us show iv) for this particular case. Indeed,

$$add(f1_A, f1_{A^c}) \;=\; \lim_n add(ff_n, f(1-f_n)) \;=\; \lim_n ff_n + f(1-f_n) \;=\; f\,.$$

We need an additional estimate for showing the general case:

8-cont (6.1) $$d_1(f1_A, 0) \;\le\; \sup_{t\in A}|f(t)| \sum_k |J_k|\,.$$

Indeed, inductively we may choose the non-overlapping J'_l in groups $J'_1, ..., J'_{l_1}$, $J'_{l_1+1}, ..., J_{l'_2}$, ... such

$$\sum_{l=l_{k-1}+1}^{l_{k+1}} |J'_l| \;\le\; |J_k|\,.$$

Then, we have

$$d_1(f1_A, 0) = \lim_n \int_0^1 |f_n f(t)|dt \;\le\; \sup_{t\in A}|f(t)| \lim_n \int_0^1 f_n(t)dt$$
$$= \sup_{t\in A}|f(t)| \sum_l |J'_l| \;\le\; \sup_{t\in A}|f(t)| \sum_k |J_k|\,.$$

Now, we consider the general case $A = \bigcup_k J_k$. We define $A_n = \bigcup_{k\le n} J_k$. We want to show that $f1_{A_n}$ is Cauchy. For this we choose non-overlapping intervals $J'_{l_{k-1}+1}, ..., J'_{l_k} \subset J_k$. Then, we deduce from (8-cont 6.1) that for $n \le m$

chy3 (6.2) $$d_1(f1_{A_n}, f1_{A_m}) \;\le\; \sup_t |f(t)| \sum_{k=n+1}^{m} \sum_{l=l_{k-1}+1}^{l_k} |J'_{l_k}| \;\le\; \sup_t |f(t)| \sum_{k=n+1}^{m} |J_k|\,.$$

Thus we may define $f1_A = \lim_n f1_{A_n}$. Again, we have

$$d_1(f1_A, g1_A) \;=\; \lim_n d_1(f1_{A_n}, g1_{A_n}) \;\le\; d_1(f,g)\,.$$

By the unique extension principle, we find a Lipschitz map $u_A : X \to X$ such that $u_A(f) = f1_A$. We use the unique extension principle to define $-x$ and the define $u_{A^c}(x) = add(x, -u_A(x))$. For $f, g \in C[0,1]$ we have

$$d_1(u_{A^c}(f), u_{A^c}(g)) \;=\; \lim_n d_1(f1_{A_n^c}, g1_{A_n^c}) \;\le\; d_1(f,g)\,.$$

Thus by unique extension this also holds for $x, y \in X$. Finally, we note that for $f \in C[0,1]$

$$add(f1_A, f1_{A^c}) \;=\; add(f1_A, add(f, -f1_A)) \;=\; \lim_n add(f1_{A_n}, add(f, -f1_{A_n}) \;=\; f \;.$$

Indeed, the equality holds for every $n \in \mathbb{N}$. Now, we will prove v). We may assume that

$$A \;=\; \bigcup_k J_k$$

such that the J_k's are non-overlapping. We define

$$A_n \;=\; \bigcup_{k \le n} J_k \;.$$

Let $\varepsilon > 0$ and $\delta > 0$ such that

$$|t - s| < \delta \quad \Rightarrow \quad |f(t) - f(s)| < \frac{\varepsilon}{2} \;.$$

Now, we consider $x \in A$ such that

$$d(x, A^c) \;=\; \inf_{y \in B} |x - y| \;\ge\; \delta$$

This means that

$$B(x, \delta) \;=\; \bigcup_k J_k \cap B(x, \delta) \;.$$

This implies

$$\lim_n |B(x, \delta) \cap A_n^c| \;=\; 0 \;.$$

Moreover, we can find a maximal family $x_1, ..., x_m$ of such points such that $d(x, A^c) \ge \delta$ implies $d(x, x_j) < \delta$ for some j. Then, we may choose n large enough such that

$$\sup_t |f(t)| \sum_{j=1}^m |B(x_j, \delta) \cap A_n^c| \;\le\; \frac{\varepsilon}{2} \;.$$

Now, we define $D = \bigcup_j \bar{B}(x_j, \delta)$. Since $A_n^c \cap D^c$ is again a collection of intervals we see that

$$\begin{aligned} d_1(f1_{A_n^c}, 0) &\le d_1(f1_{A_n^c \cap D}, 0) + d_1(f1_{A_n^c \cap D^c}, 0) \\ &\le \sup_t |f(t)| \sum_{j=1}^m |B(x_j, \frac{\delta}{2}) \cap A_n^c| + \sup_{t \in A_n^c \cap D^c} |f(t)| \;. \end{aligned}$$

Now, we consider $t \in A_n^c \cap D^c$. If $d(t, B) \ge \frac{\delta}{2}$, then $d(t, x_j) \le \frac{\delta}{2}$ hence $t \in D$. Thus we may assume $d(t, B) < \frac{\delta}{2}$. Then we find $s \in B$ such that $|t - s| < \delta$ and thus

$$|f(t)| \;\le\; \sup_{s \in B} |f(s)| + \varepsilon \;.$$

This implies

$$d_1(f1_{A_n^c},0) \;\le\; \frac{\varepsilon}{2}+\varepsilon+\sup_{s\in B}|f(s)|\,.$$

for $n \ge n_0$. The assertion is proved. ∎

REMARK 6.8. *For $J=[a,b]$ we have*

$$I(f1_{[a,b]}) \;=\; \int_a^b f(t)dtpl.$$

approx

LEMMA 6.9. *Let (A_m) be a sequence such that*

$$A_m \;=\; \bigcup_k J_k^m$$

and

$$\lim_m \sum_k |J_k^m| \;=\; 0\,.$$

Then

$$\lim_m d_1(x1_{A_m},0) \;=\; 0$$

holds for every $x \in X$.

PROOF. Let (f_n) be Cauchy sequence converging to x. Let $\varepsilon > 0$. Then, we may choose n such that

$$d_1(f_n,x) < \frac{\varepsilon}{2}\,.$$

Since u_{A_m} is Lipschitz, we deduce

$$d_1(f_n1_{A_m},u_{A_m}(x)) < \frac{\varepsilon}{2}\,.$$

for all $m \in \mathbb{N}$. According to (8-cont 6.1) we find

$$d_1(f_n1_{A_m},0) \;\le\; \sup_t |f_n(t)| \sum_k |J_k^m|\,.$$

By assumption, we may find m_0 such that

$$d_1(f_n1_{A_m},0) < \frac{\varepsilon}{2}$$

holds for all $m \ge m_0$. This implies

$$d_1(u_{A_m}(x),0) \;\le\; d_1(u_{A_m}(x),f_n1_{A_m}) + d_1(f_n1_{A_m},0) < \varepsilon$$

for all $m > m_0$. ∎

Proof of Theorem 6.6. Let (f_n), (g_n) be Cauchy such that $d_1-\lim_n f_n = x$ and $d_1-\lim_n g_n = y$ and f_n converges to f and g_n converges to g a.e. We consider $h_n' = f_n - g_n$ and $z \;=\; add(x,-y)$. We want to show

$$d_1(x,0) \;=\; 0\,.$$

Clearly, h_n' converges to 0 almost everywhere. Passing to subsequence (h_k) we may assume that $d_1(h_k,h_{k+1}) \;\le\; 6^{-n}$. According to the proof of Proposition 6.5, we find $A_k = \bigcup_l J_l^k$ such that

$$\sum_l |J_k^l| \;\le\; 2^{1-k}$$

such that

$$|h_n(t)-h_{n+1}(t)| \;\le\; 2^{-n}$$

for all $t \notin A_k$. By the definition of a.e. we find $B_k = \bigcup_l \tilde{J}_l^k$ such that

$$\sum_l |\tilde{J}_k^l| \;\le\; 2^{1-k}$$

and

$$\lim_t h_n(t) \;=\; 0$$

holds for all $t \notin B_k$. Thus we define

$$C_k \;=\; \bigcup_l J_k^l \cup \bigcup_l \tilde{J}_k^l\,.$$

Then

$$\sum_l |J_k^l| + \sum_l |\tilde{J}_k^l| \;\le\; 2^{2-k}$$

and for all $t \notin C_k$ we have

$$|h_n(t)| \;=\; |\lim_m h_n(t)-h_m(t)| \;\le\; \limsup_m |h_n(t)-h_m(t)| \;\le\; 2^{1-n}\,.$$

According to Lemma (6.9) we find a k_0 such that

$$d_1(u_{C_k}(z),0) < \varepsilon$$

for all $k \ge k_0$. For all $n \ge k$ we deduce from Lemma 6.7 that

$$d_1(1_{C_k^c} f_n, 0) \;\le\; 2^{1-n}\,.$$

Therefore

$$d_1(u_{C_k^c}(z),0) \;=\; \lim_n d_1(u_{C_k^c}(f_n),0) \;=\; 0\,.$$

Since $x = add(u_{C_k^c}(x), u_{C_k^c}(x))$, we deduce

$$d_1(z,0) = d_1(add(u_{C_k}(z),u_{C_k^c}(z)), add(u_{C_k}(0),u_{C_k^c}(0)))$$

$$\leq d_1(u_{C_k}(z), 0) + d_1(u_{C_k^c}(z), 0) < \varepsilon \; .$$

Finally, we observe that

$$d_1(x, y) \;=\; d_1(add(x, -y), 0)$$

holds by unique extension. Thus $x = y$. ■

7. Closed and Compact Sets

Let (X,d) be a metric space. We will say that a subset $A \subset X$ is *closed* if $X \setminus A$ is open.

closed PROPOSITION 7.1. *Let (X,d) be a complete metric space and $C \subset X$ a subset. C is closed iff every Cauchy sequence in C converges to an element in C.*

Proof: Let us assume C is closed and that (x_n) is a Cauchy sequence with elements in C. Let $x = \lim_n x_n$ be te limit and assume $x \notin C$. Since $X \setminus C$ is open

$$B(x,\varepsilon) \subset X \setminus C$$

for some $\varepsilon > 0$. Then there exists an n_0 such that $d(x_n, x) < \varepsilon$ for $n > n_0$. In particular,

$$x_{n_0+1} \in B(x,\varepsilon)$$

and thus $x_{n_0+1} \notin C$, a contradiction.
Now, we assume that every Cauchy sequence with values in C converges to an element in C. If $X \setminus C$ is not open, then there exists an $x \notin C$ and no $\varepsilon > 0$ such that

$$B(x,\varepsilon) \subset X \setminus C\,.$$

I.e. for every $n \in \mathbb{N}$, we can find $x_n \in C$ such that

$$d(x, x_n) < \frac{1}{n}\,.$$

Hence, $\lim x_n = x \in C$ but $x \notin C$, contradiction. ■
The most important notion in this class is the notion of compact sets. We will say that a subset $C \subset X$ is *compact* if For every collection (O_i) of open sets such that

$$C \subset \bigcup_i O_i \;=\; \{x \in X \mid \exists_{i \in I} x \in O_i\}$$

There exists $n \in \mathbb{N}$ and $i_1, ..., i_n$ such that

$$C \subset O_{i_1} \cup \cdots \cup O_{i_n}\,.$$

In other words

Every open cover of C has a finite subcover .

DEFINITION 7.2. *Let $X \subset \bigcup O_i$ be an open cover. Then we say that (V_j) is an open subcover if*

$$X \subset \bigcup_j V_j$$

all the V_j are open and for every j there exists an i such that

$$V_j \subset O_i \, .$$

It is impossible to explain the importance of 'compactness' right away. But we can say that there would be no discipline 'Analysis' without compactness. The most clarifying idea is contained in the following example.

PROPOSITION 7.3. *The set $[0,1] \subset \mathbb{R}$ is compact.*

Proof: Let $[0,1] \subset \bigcup_i O_i$. For every $x \in [0,1]$ there exists an $i \in I$ such that

$$x \in O_i \, .$$

Since O_i is open, we can find $\varepsilon > 0$ such that

$$x \in B(x,\varepsilon) \subset O_i \, .$$

Using the axiom of choice, we fine a function ε_x and i_x such that

$$x \in B(x,\varepsilon_x) \subset O_{i_x} \, .$$

Let us define the relation $x \preceq y$ if $x < y$ and

$$y - x \;\le\; e_x + e_y \, .$$

The crucial point here is to define

$$S \;=\; \{x \in [0,1] \mid \exists\, x_1, ..., x_n \;: \frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq x\} \, .$$

We claim a) $\sup S \in S$ and b) $\sup S = 1$.

Ad a): Let $y = \sup S \in [0,1]$. Then there exists an $x \in S$ with

$$y - \varepsilon_y < x \le y \, .$$

Then obviously $x \preceq y$. Since $x \in S$, we can find

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq x \preceq y \, .$$

Thus $y \in S$.

Ad b): Assume $y = \sup S < 1$. Let $0 < \delta \;=\; \min(e_y, 1-y)$. Then

$$y + \delta - y \;=\; \delta \;\le\; \varepsilon_y + \varepsilon_{y+\delta} \, .$$

By a), we find

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq y \preceq y + \delta$$

and thus $y + \delta \in S$. Contradiction to the definition of the supremum. Assertion a) and b) are proved.

Therefore we conclude $1 \in S$ and thus find $x_1, ..., x_n$ such that

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq 1 \, .$$

Let $x_0 = \frac{1}{2}$ and $x_{n+1} = 1$, then by definition of $\preceq$, we have

$$[x_j, x_{j+1}] \subset B(x_j, \varepsilon_{x_j}) \cup B(x_{j+1}, \varepsilon_{x_{j+1}}) \subset O_{i_{x_j}} \cup O_{i_{x_{j+1}}}$$

for $j = 0, .., n$. Thus, we deduce

$$[\frac{1}{2}, 1] \subset \bigcup_{j=0}^{n} [x_j, x_{j+1}] \subset \bigcup_{j=0}^{n+1} O_{i_{x_j}} \, .$$

Doing the same trick with $[0, \frac{1}{2}]$, we find

$$[0, 1] \subset \bigcup_{j=0}^{m+1} O_{i_{x'_j}} \cup \bigcup_{j=0}^{n+1} O_{i_{x_j}}$$

and we have found our finite subcover. ■

subcom PROPOSITION 7.4. *Let $B \subset X$ be closed set and $C \subset X$ be a compact set, then*

$$B \cap C$$

is compact

Proof: Let $B \cap C \subset \bigcup O_i$ be an open cover. then

$$C \subset (X \setminus B) \cup \bigcup_i O_i$$

is an open cover for C, hence we can find a finite subcover

$$C \subset (X \setminus B) \cup O_{i_1} \cup \cdots \cup O_{i_n} \, .$$

Thus

$$B \cap C \ \subset O_{i_1} \cup \cdots \cup O_{i_n}$$

is a finite subcover. ■

ccc LEMMA 7.5. *Let (X, d) be a metric space and $D \subset X$ be a countable dense set in X, then for every subset $C \subset X$ and every open cover*

$$C \subset \bigcup_i O_i$$

we can find a countable subcover of balls.

Proof: Let us enumerate D as $D = \{d_n \,|\, n \in \mathbb{N}\}$. Let $x \in C$ and find $i \in I$ and $\varepsilon > 0$ such that

$$x \in B(x, \varepsilon) \subset O_i \, .$$

Let $k > \frac{2}{\varepsilon}$. By density, we can find an $n \in \mathbb{N}$ such that

$$d(x, d_n) < \frac{1}{2k} \, .$$

Then

$$x \in B(d_n, \frac{1}{2k}) \subset B(x, \frac{1}{k}) \subset B(x, \varepsilon) \subset O_i \, .$$

Let us define

$$M \;=\; \{(n, k) \mid \exists_{i \in I} B(d_n, \frac{1}{2k}) \subset O_i\} \, .$$

Then $M \subset \mathbb{N}^2$ is countable and hence there exists a map $\phi : \mathbb{N} \to M$ which is surjective (=onto). Hence for $V_m = B(d_{\phi_1(m)}, \frac{1}{2\phi_2(m)})$, ϕ_1, ϕ_2 the 2 components of ϕ we have

$$C \subset \bigcup_m V_m$$

and (V_m) is a countable subcover of balls of the original cover (O_i). ■

main THEOREM 7.6. *Let (X, d) be a metric space. Let $C \subset X$ be a subset. Then the following are equivalent*

i) a) *Every Cauchy sequence of elements in C converges to a limit in C.*
b) *For every $\varepsilon > 0$ there exists points $x_1, ..., x_n \in X$ such that*

$$C \subset B(x_1, \varepsilon) \cup \cdots \cup B(x, \varepsilon) \, .$$

ii) *Every sequence in C has a convergent subsequence.*
iii) *C is compact.*

Proof: *i)* $\Rightarrow$ *ii)*. Let (x_n) be a sequence. Inductively, we will construct infinite subset $A_1 \supset A_2 \supset A_3 \cdots$ and y_1, y_2, y_3,... in X such that

$$\forall_{l \in A_j} \; : \; d(x_l, y_j) < 2^{-j-1} \, .$$

Put $A_0 = \mathbb{N}$. Let us assume $A_1 \supset A_2 \supset \cdots A_n$ and $y_1, ..., y_n$ have been constructed. We put $\varepsilon = 2^{-n-2}$ and apply condition $i)b)$ to find $z_1, ..., z_m$ such that

$$C \;\subset\; B(z_1, \varepsilon) \cup \cdots \cup B(z_m, \varepsilon) \,.$$

We claim that there must be a $1 \leq k \leq m$ such that

$$A_n(k) \;=\; \{l \in A_n \,|\, x_n \in B(z_k, \varepsilon)\}$$

has infinitely many elements. Indeed, we have

$$A_n(1) \cup \cdots \cup A_n(m) \;=\; A_n \,.$$

If they were all finite, then a finite union of finite sets would have finitely many elements. However A_n is infinite. Contradiction! Thus, we can find a k with $A_n(k)$ infinite and put $A_{n+1} = A_n(k)$ and $y_{n+1} = z_k$. So the inductive procedure is finished. Now, we can find an increasing sequence (n_j) such that $n_j \in A_j$ and deduce

$$d(x_{n_j}, x_{n_{j+1}}) \;\leq\; d(x_{n_j}, y_j) + d(y_j, x_{n_{j+1}}) < \frac{1}{2}2^{-j} + \frac{1}{2}2^{-j} \;=\; 2^{-j}$$

because $n_j \in A_j$ and $n_{j+1} \in A_{j+1} \subset A_j$. Thus (x_{n_j}) is Cauchy. Indeed, be induction, we deduce for $j < m$ that

$$\begin{aligned} d(x_{n_j}, x_{n_m}) \;&\leq\; d(x_{n_j}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_{n_{j+2}}) \cdots d(x_{n_{m-1}}, x_{n_m}) \\ &\leq\; 2^{-j} \sum_{k=0}^{m-1} 2^{-k} \;=\; 2^{1-j} \,. \end{aligned}$$

This easily implies the Cauchy sequence condition. By a) it converges to some $x \in C$. We got our convergent subsequence.

$ii) \Rightarrow iii)$: We will first show $ii) \Rightarrow i)b)$. Indeed, let $\varepsilon > 0$ and assume for all $n \in \mathbb{N}$, $y_1, ..., y_n \in C$ we may find

$$x(n, y_1, ..., y_n) \in C \setminus (B(y_1, \varepsilon) \cup \cdots B(y_n, \varepsilon)) \,.$$

Then we define $x_1 \in C$ and find $x_2 \in C \setminus B(x_1, \varepsilon)$. Then we find $x_3 \in C \setminus B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$. Thus inductively we find $x_n \in C$ such that

$$d(x_n, x_k) \geq \varepsilon$$

for all $1 \leq k \leq n$. It is easily seen that (x_n) has no convergent subsequence. Thus $i)b)$ is showed (with points in C). For every $\varepsilon_k = \frac{1}{k}$ we find these points $y_1^k, ..., y_{m(k)}^k \in C$ such that

$$C \subset B(y_1^k, \frac{1}{k}) \cup \cdots \cup B(y^{m(k)}, \frac{1}{k}) \,.$$

Then, we see that $D = \{y_j^k : k \in \mathbb{N}, 1 \le j \le m(k)\}$ is dense in C. Therefore, we may work with the closure $\tilde{X} = \bar{D}$ and show that C is compact in $\tilde{X}$. (It will then be automatically compact in X). By Lemma 7.5, we may assume that

$$C \subset \bigcup_k O_k$$

and O_k's open. If we can find an n such that

$$C \subset O_1 \cup \cdots \cup O_n$$

the assertion is proved. Assume that is not the case and choose for every $n \in \mathbb{N}$ an $x_n \in C \setminus O_1 \cup \cdots \cup O_n$. According to the assumption, we have a convergent subsequence, i.e. $\lim_k x_{n_k} = x \in C$. Then $x \in O_{n_0}$ for some n_0 and there exists a $\varepsilon > 0$ such that

$$B(x, \varepsilon) \ \subset O_{n_0} \, .$$

By convergence, we find a k_0 such that $d(x, x_{n_k}) < \varepsilon$ for all $k > k_0$. In particular, we find a $k > k_0$ such that $n_k > n_0$. Thus

$$x_{n_k} \in B(x, \varepsilon) \in O_{n_0} \subset O_1 \cup \cdots \cup O_{n_k} \, .$$

Contradicting the choice of the (x_n)'s. We are done.
iii) $\Rightarrow$ *i)b)* Let $\varepsilon > 0$ and then

$$C \ \subset \bigcup_{x \in C} B(x, \varepsilon) \, .$$

thus a finite subcover yields *b)*.
iii) $\Rightarrow$ *i)a)* Let (x_n) be a Cauchy sequence. Assume it is not converging to some element $x \in C$. This means

cccc (7.1)
$$\forall x \in C \exists \varepsilon(x) > 0 \forall n_0 \exists n > n_0 \; d(x_n, x) > \varepsilon \, .$$

Then

$$C \subset \bigcup_{x \in C} B(x, \frac{\varepsilon(x)}{2}) \, .$$

Let

$$C \subset B(y_1, \frac{\varepsilon(y_1)}{2}) \cup \cdots \cup B(y_1, \frac{\varepsilon(y_1)}{2})$$

be a finite subcover (compactness). Then there exists at least one $1 \le k \le m$ such that

$$A_k \ = \ \{n \in \mathbb{N} \mid d(x_n, y_k) < \frac{\varepsilon(y_k)}{2}\}$$

is infinite. Fix that k and apply the Cauchy criterion to find n_0 such that

$$d(x_n, x_{n'}) < \frac{\varepsilon(y_k)}{2}$$

for all $n, n' > n_0$. By (cccc 7.1), we can find an $n > n_0$ such that

$$d(x_n, y_k) > \varepsilon(y_k)\ .$$

Since A_k is infinite, we can find an $n' > n_0$ in A_k thus

$$\begin{aligned} \varepsilon(y_k) \ &< \ d(x_{n'}, y_k) \ \le \ d(x_n, x_{n'}) + d(x_{n'}, y_k) \\ &< \ \frac{\varepsilon(y_k)}{2} + \frac{\varepsilon(y_k)}{2} \ = \ \varepsilon(y_k)\ . \end{aligned}$$

A contradiction. Thus the Cauchy sequence has to converge to some point in C. ■

Corollary 7.7. *Every intervall $[a,b] \subset \mathbb{R}$ with $a < b \in \mathbb{R}$ is compact*

Proof: It is easy to see that $X \setminus [a,b]$ is open. Hence, by Proposition closed 7.1 $[a,b]$ is complete, i.e. i)a) is satisfied. Given $\varepsilon > 0$, we can find $k > \frac{1}{\varepsilon}$. For $m > k(b-a)$ we derive

$$[a,b] \subset \bigcup_{j=0}^{m} B(a + \frac{j}{k}, \varepsilon)\ .$$

Thus the Theorem main 7.6 applies. ■

cube Lemma 7.8. *Let $r > 0$ and $n \in \mathbb{N}$, the set $C_r = [-r,r]^n$ is compact.*

Proof: Let $x \notin C_r$, then there exists an index $j \in \{1,..,n\}$ such that $|x_j| > r$. Let $\varepsilon = |x_j| - r$ and $y \in \mathbb{R}^n$ such that

$$\max_{i=1,..,n} |x_i - y_i| < \varepsilon\ ,$$

then

$$|y_j| \ = \ |y_j - x_j + x_j| \ \ge \ |x_j| - |y_j - x_j| \ > \ |x_j| - \varepsilon \ = \ r\ .$$

thus $y \notin C_r$. Hence, C_r is closed and according to Proposition rncomp 4.3, we deduce that C_r is complete.

For $n = 1$ and $\varepsilon > 0$, we have seen above that for $k > \frac{1}{\varepsilon}$ and $m > \frac{2r}{k}$

$$[-r,r] \subset \bigcup_{j=0}^{m} B(-r + \frac{j}{k}, \varepsilon)\ .$$

Therefore

$$[-r,r]^n \subset \bigcup_{j_1,...,j_n=0,...m} B_\infty((-r + \frac{j_1}{k}, .., -r + \frac{j_n}{k}), \varepsilon)\ .$$

Thus i)a) and i)b) are satisfies and the Theorem main 7.6 implies the assertion (The separable dense subset is $\mathbb{Q}^n$.) ■

THEOREM 7.9. *Let $C \subset \mathbb{R}^n$ be a subset. The following are equivalent*

1) *C is compact.*
2) *C is closed and there exists an r such that*

$$C \subset B(0, R) .$$

(That is C is bounded.)

Proof: 2) $\Rightarrow$ 1) Let

$$C \subset B(0, R) \subset [-R, R]^n$$

be a closed set. Since $[-R, R]^n$ is compact, we deduce from Proposition 7.4 that C is compact as well.

1) $\Rightarrow$ 2) Let C subset $\mathbb{R}^n$ be a compact set. According to Theorem 7.6 i)b), we find

$$C \subset B(x_1, 1) \cup \cdots \cup B(x_m, 1)$$

thus for $r = \max_{i=1,..,m}(d(x_i, 0) + 1)$ we have

$$C \subset B(0, r) .$$

Moreover, by Theorem 7.6 i)a) and Proposition 7.1, we deduce that C is closed. ■

We will now discuss one of the most important applications.

THEOREM 7.10. *Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. The there exists $x_0 \in X$ such that*

$$f(x_0) \ = \ \sup\{f(x) : x \in X\} .$$

PROOF. Let us first assume

$$A = \{f(x) : x \in X\}$$

is bounded and $s = \sup A$. For every $n \in \mathbb{N}$, we know that $s - \frac{1}{n}$ is no upper bound. Hence there $x_n \in X$ such that

$$s \geq f(x_n) > s - \frac{1}{n} .$$

Let (n_k) be such that $\lim_k x_{n_k} = x \in X$. Then we deduce from continuity that

$$f(x) \ = \ \lim_k f(x_{n_k}) \ \geq \ \lim s - \frac{1}{n_k} \ = \ s .$$

By definition of s we find $f(x) = s$. Now, we show that A is bounded. Indeed, if note we find $x_n \in X$ such that $f(x_n) \geq n$. Again we find a convergent subsequence

(x_{n_k}). Since $f(x_{n_k})$ is convergent it is bounded. We assume (f_{n_k}) is bounded above by $m \in nz$. Choosing $k \geq m+1$ we get

$$m \geq f(x_{n_k}) \geq n_k > n_m \geq m \, .$$

This contradiction shows that A is bounded and hence the first argument applies. ■