1

Real Numbers and Monotone Sequences

1.1 Introduction. Real numbers.

Mathematical analysis depends on the properties of the set $\mathbb{R}$ of real numbers, so we should begin by saying something about it.

There are two familiar ways to represent real numbers. Geometrically, they may be pictured as the points on a line, once the two reference points corresponding to 0 and 1 have been picked. For computation, however, we represent a real number as an infinite decimal, consisting of an integer part, followed by infinitely many decimal places:

$$3.14159\ldots, \qquad -.033333\ldots, \qquad 101.2300000\ldots\ .$$

There are difficulties with decimal representation which we need to think about. The first is that two different infinite decimals can represent the same real number, for according to well-known rules, a decimal having only 9's after some place represents the same real number as a different decimal ending with all 0's (we call such decimals *finite* or *terminating*):

$$26.67999\ldots = 26.68000\ldots = 26.68\ , \qquad -99.999\ldots = -100.$$

This ambiguity is a serious inconvenience in working theoretically with decimals.

> Notice that when we write a finite decimal, in mathematics the infinite string of decimal place zeros is dropped, whereas in scientific work, some zeros are retained to indicate how accurately the number has been determined.

Another difficulty with infinite decimals is that it is not immediately obvious how to calculate with them. For finite decimals there is no problem; we just follow the usual rules—add or multiply starting at the right-hand end:

$$\begin{array}{r} 2.389 \\ +\ \underline{2.389} \\ \ldots 78 \end{array} \qquad\qquad \begin{array}{r} 2.849 \\ \times\ \underline{\quad .09} \\ \ldots 41 \end{array}$$

But an infinite decimal has no right-hand end...

To get around this, instead of calculating with the infinite decimal, we use its truncations to finite decimals, viewing these as approximations to the infinite decimal. For instance, the increasing sequence of finite decimals

(1) $$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

gives ever closer approximations to the infinite decimal $\pi = 3.1415926\ldots$; we say that π is the *limit* of this sequence (a definition of "limit" will come soon).

To see how this allows us to calculate with infinite decimals, suppose for instance we want to calculate

$$\pi + \sqrt[3]{2}\ .$$

We write the sequences of finite decimals which approximate these two numbers:

$$\begin{array}{llllllll} \pi & \text{is the limit of} & 3, & 3.1, & 3.14, & 3.141, & 3.1415, & 3.14159, \ldots; \\ \sqrt[3]{2} & \text{is the limit of} & 1, & 1.2, & 1.25, & 1.259, & 1.2599, & 1.25992, \ldots; \end{array}$$

then we add together the successive decimal approximations:

$$\begin{array}{llllllll} \pi+\sqrt[3]{2} & \text{is the limit of} & 4, & 4.3, & 4.39, & 4.400, & 4.4014, & 4.40151, \ldots, \end{array}$$

obtaining a sequence of numbers which also increases.

The decimal representation of this increase isn't as simple as it was for the sequence representing π, since as each new decimal digit is added on, the earlier ones may change. For instance, in the fourth step of the last row, the first decimal place changes from 3 to 4. Nonetheless, as we compute to more and more places, the earlier part of the decimals in this sequence ultimately doesn't change any more, and in this way we get the decimal expansion of a new number; we then define the sum $\pi + \sqrt[3]{2}$ to be this number, $4.4015137\ldots$.

We can define multiplication the same way. To get $\pi \times \sqrt[3]{2}$, for example, multiply the two sequences above for these numbers, getting the sequence

$$3, \quad 3.72, \quad 3.9250, \quad 3.954519, \quad \ldots \ . \tag{2}$$

Here too as we use more decimal places in the computation, the earlier part of the numbers in the sequence (2) ultimately stops changing, and we define the number $\pi \times \sqrt[3]{2}$ to be the limit of the sequence (2).

As the above shows, even the simplest operations with real numbers require an understanding of sequences and their limits. These appear in analysis whenever you get an answer not at once, but rather by making closer and closer approximations to it. Since they give a quick insight into some of the most important ideas in analysis, they will be our starting point, beginning with the sequences whose terms keep increasing (as in (1) and (2) above), or keep decreasing. In some ways these are simpler than other types of sequences.

Appendix A.0 contains a brief review of set notation, and also describes the most essential things about the different number systems we will be using: the integers, rational numbers, and real numbers, as well as their relation to each other. Look through it now just to make sure you know these things.

Questions 1.1

(Answers to the Questions for each section of this book can be found at the end of the corresponding chapter.)

1. In the sequence above for $\pi + \sqrt[3]{2}$, the first decimal place of the final answer is not correct until four steps have been performed. Give an example of addition where the first decimal place of the final answer is not correct until k steps have been performed. (Here k is a given positive integer.)

1.2 Increasing sequences.

By a **sequence** of numbers, we mean an infinite list of numbers, written in a definite order so that there is a first, a second, and so on; we write it either

(3) $$a_0,\ a_1,\ a_2,\ a_3, \ldots, a_n, \ldots, \qquad \text{or} \quad \{a_n\},\ n \geq 0.$$

We call a_n the **n-th term** of the sequence; often there is an expression in n for it. Some simple examples of sequences written in both forms are:

(4) $$1,\ 1/2,\ 1/3,\ 1/4,\ \ldots \qquad \{1/n\},\ n \geq 1$$

(5) $$1,\ -1,\ 1,\ -1,\ \ldots \qquad \{(-1)^n\},\ n \geq 0$$

(6) $$1,\ 4,\ 9,\ 16,\ \ldots \qquad \{n^2\},\ n \geq 1$$

(7) $$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ \ldots$$

For the last sequence, there is no expression in n for the n-th term. In the other cases, the range of values of n is specified, though this can be omitted if it is the standard choice $n \geq 0$. As this book progresses, we will with increasing frequency omit the braces, referring to (5) for example simply as the sequence $(-1)^n$.

Definition 1.2 We say the sequence $\{a_n\}$ is

increasing if $a_n \leq a_{n+1}$ for all n; **strictly increasing** if $a_n < a_{n+1}$ for all n;
decreasing if $a_n \geq a_{n+1}$ for all n; **strictly decreasing** if $a_n > a_{n+1}$ for all n.

As examples, the sequence (4) is strictly decreasing, (6) is strictly increasing, while (7) is only increasing since zeros occur in the infinite decimal for π. The phrase "for all n" has the meaning "for all values of n for which a_n is defined"; this is usually $n \geq 0$ or $n \geq 1$ for the sequences in this chapter.

According to the definition, the sequence $2, 2, 2, \ldots$ has to be called increasing. This may seem strange, but remember that, like Humpty-Dumpty, mathematicians can define words to mean whatever they want them to mean. Here the mathematical world itself is split over what one should call these sequences. One possibility is on the right, but our choice is on the left — we have a dislike for negative-sounding words, since they point you in the non-right direction.

increasing	=	non-decreasing;
strictly increasing	=	increasing.

Questions 1.2 (Answers at end of chapter)

1. Under the natural ordering, which of the following are sequences?

   (a) all integers (b) all integers ≥ -100 (c) all integers ≤ 0

2. Give each sequence in the form $\{a_n\}$, $n \geq \ldots$, as in (4) or (5):

   (a) $0, 1, 0, -1, 0, 1, 0, -1, \ldots$ (use $\sin x$) (b) $1/2,\ 2/3,\ 3/4, \ldots$

3. For each of the following sequences, tell without proof whether it is increasing (strictly?), decreasing (strictly?) or neither.

(a) $\{(3/4)^n\},\ n \geq 0$ (b) $\{\cos(1/n)\},\ n \geq 1$ (c) $\left\{\dfrac{n-1}{n}\right\},\ n \geq 1$

(d) $\{n^2 - n\},\ n \geq 0$ (e) $\{n(n-2)\},\ n \geq 0$ (f) $\{\ln(1/n)\},\ n \geq 1$

1.3 The limit of an increasing sequence.

We now make our earlier observations about adding and multiplying reals more precise by giving a provisional definition for the limit of an increasing sequence. (A more widely applicable definition will be given in Chapter 3.)

In the definition, we assume for definiteness that none of the a_n ends with all 9's—i.e., they are written as terminating decimals, if possible. The limit L however might appear in either form (cf. Question 1.3/3 below); we will refer to the form in which it appears as a "suitable" decimal representation for L.

Definition 1.3A A number L, in a suitable decimal representation, is the **limit** of the increasing sequence $\{a_n\}$ if, given any integer $k > 0$, all the a_n after some place in the sequence agree with L to k decimal places.

The two notations for limit are (often the braces are omitted):

$$\lim_{n\to\infty} \{a_n\} = L, \qquad \{a_n\} \to L \text{ as } n \to \infty.$$

If such an L exists, it must be unique, since its first k decimal places (for any given k) are the same as those of all the a_n sufficiently far out in the sequence.

On the other hand, such an L need not exist; the sequence $1, 2, \ldots, n, \ldots$ has no limit, for example. Here is the key hypothesis which is needed.

Definition 1.3B A sequence $\{a_n\}$ is said to be **bounded above** if there is a number B such that $a_n \leq B$ for all n.

Any such B is called an **upper bound** for the sequence.

For example, the sequences (4), (5), and (7) are bounded above, while (6) is not. For (4) and (5), any number ≥ 1 is an upper bound.

Theorem 1.3

A positive increasing sequence $\{a_n\}$ which is bounded above has a limit.

We cannot give a formal proof but hope the argument below will seem plausible to those who have watched odometers on long car trips. (The theorem is also true for sequences with negative terms; these will be discussed in Section 1.6.)

Write out the decimal expansions of the numbers a_n and arrange them in a list, as illustrated at right.

$$\begin{aligned}
a_0 &= 15.34576\ldots\\
a_1 &= 16.26745\ldots\\
a_2 &= 16.33654\ldots\\
a_3 &= 16.34722\ldots\\
a_4 &= 16.34745\ldots\\
a_5 &= 16.34747\ldots\\
a_6 &= 16.34748\ldots
\end{aligned}$$

Look down the list of numbers. We claim that after a while the integer part and first k decimal places of the numbers on the list no longer change. Take these unchanging values to be the corresponding places of the decimal expansion of the limit L.

To see this in more detail, look first at the integer parts of the numbers in the list. They increase (in the sense of Definition 1.2), but they cannot strictly increase infinitely often, because the sequence formed by the integer parts is bounded above. So after some index $n = n_0$, the integer part never changes.

Starting from this term a_{n_0}, continue down the list, looking now just at the first decimal place. It increases (Definition 1.2), but if it ever got beyond 9, i.e., turned into 0, the integer part would have to change, and we just agreed it doesn't. So after some later index $n_1 \geq n_0$, the first decimal place will stay constant.

Continue down from the term a_{n_1}; after a while the second decimal place will stay constant, otherwise it would get beyond 9 and the first decimal place would have to change. Continuing in this way (or using mathematical induction — see Appendix A.4), we see that ultimately the integer part and first k decimal places remain constant, and these define the first k decimal places of L. Since k was arbitrary, we have defined L. □

Questions 1.3

1. Which of these sequences is bounded above? For each that is, give an upper bound. (In each case use $n \geq 0$ if it makes sense, otherwise $n \geq 1$.)

(a) $\{(-1)^n/n\}$ (b) $\{\sqrt{n}\}$ (c) $\{\sin n\}$ (d) $\{\ln n\}$

2. Which of these increasing sequences is bounded above? For each that is, give: (i) an upper bound; (ii) the limit.

(a) $a_n = (n-1)/n,\ n \geq 1$ (b) $a_n = \cos(1/n),\ n \geq 1$

(c) $a_n = 2n/(n+1)$ (d) $a_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$

3. Apply the method given in the argument for Theorem 1.3 to find the "suitable" decimal representation (cf. Definition 1.3A) of the limit L of the increasing sequence $a_n = 1 - 1/10^n$.

4. Where in the plausibility argument are we using the fact that the a_n are written in terminating form, if possible?

1.4 Example: the number e

We saw in Section 1.1 how the notion of limit lets us define addition and multiplication of positive real numbers. But it also gives us an important and powerful method for constructing particular real numbers. This section and the next give examples. They require some serious analytic thinking and give us our first proofs.

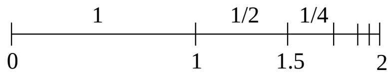
The aim in each proof is to present an uncluttered, clear, and convincing argument based upon what most readers already know or should be willing to accept as clearly true. The first proof for example refers explicitly to the binomial theorem

$$(8)\qquad (1+x)^k = 1+kx+\ldots+\binom{k}{i}x^i+\ldots+x^n, \qquad \binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!}\ ,$$

which you should know. But it also uses without comment the result

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n} \ < \ 2,$$

which is "obvious" geometrically:



and also follows from the formula for the geometric sum (taking $r = 1/2$):

$$1+r+r^2+\ldots+r^n = \frac{1-r^{n+1}}{1-r}\ .$$

If you didn't think of the picture and didn't remember or think of using the formula, you will feel a step has been skipped. One person's meat is another person's gristle; just keep chewing and it will ultimately go down.

As motivation for this first example, we recall the compound interest formula: invest P dollars at the annual interest rate r, with the interest compounded at equal time intervals n times a year; by the end of the year it grows to the amount

$$A_n = P\left(1+\frac{r}{n}\right)^n\ .$$

Thus if we invest one dollar at the rate $r = 1$ (i.e., 100% annual interest), and we keep recalculating the amount at the end of the year, each time doubling the frequency of compounding, we get a sequence beginning with

$$\begin{array}{lll} A_1 = 1+1 & = \ 2 & \textit{simple interest;} \\ A_2 = (1+1/2)^2 & = \ 2.25 & \textit{compounded semiannually;} \\ A_4 = (1+1/4)^4 & \approx \ 2.44 & \textit{compounded quarterly.} \end{array}$$

Folk wisdom suggests that successive doubling of the frequency should steadily increase the amount at year's end, but within bounds, since banks do manage to stay in business even when offering daily compounding. This should make the following proposition plausible. (The limit is e.)

Proposition 1.4 *The sequence* $a_n = \left(1+\dfrac{1}{2^n}\right)^{2^n}$ *has a limit.*

Proof.

By Theorem 1.3, it suffices to prove $\{a_n\}$ is increasing and bounded above.

To show it is increasing, if $b \neq 0$ we have $b^2 > 0$, and therefore,

$$(1+b)^2 \ > \ 1+2b\ ;$$

raising both sides to the 2^n power, we get

$$(1+b)^{2\cdot 2^n} \ > \ (1+2b)^{2^n}\ .$$

If we now put $b = 1/2^{n+1}$, this last inequality becomes $a_{n+1} > a_n$. □

To show that a_n is bounded above, we will prove a stronger statement ("stronger" because it implies that a_n is bounded above: cf. Appendix A):

$$(9) \qquad \left(1+\frac{1}{k}\right)^k \le 3 \qquad \text{for any integer } k \ge 1 .$$

To see this, we have by the binomial theorem (8),

$$(10) \quad \left(1+\frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \ldots + \frac{k(k-1)\cdots(k-i+1)}{i!}\left(\frac{1}{k}\right)^i + \ldots + \frac{k!}{k!}\left(\frac{1}{k}\right)^k .$$

To estimate the terms in the sum on the right, we note that

$$k(k-1)\cdots(k-i+1) \le k^i , \qquad i = 1, \ldots, k ,$$

since there are i factors on the left, each at most k; and by similar reasoning,

$$(11) \qquad \frac{1}{i!} = \frac{1}{i}\cdot\frac{1}{i-1}\cdot \;\cdots\; \cdot\frac{1}{2} \le \left(\frac{1}{2}\right)^{i-1}, \quad i = 2, \ldots, k .$$

Therefore, for $i = 2, \ldots, k$ (and $i = 1$ also, as you can check),

$$(12) \qquad \frac{k(k-1)\cdots(k-i+1)}{i!}\cdot\left(\frac{1}{k}\right)^i \le \frac{1}{2^{i-1}} .$$

Using (12) to estimate the terms on the right in (10), we get, for $k \ge 2$,

$$(13) \qquad \begin{aligned} \left(1+\frac{1}{k}\right)^k &\le 1 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{k-1}} ; \\ &\le 1 + 2 ; \end{aligned}$$

and this is true for $k = 1$ as well. □□

Remarks.

1. Euler was the first to encounter the number $\lim a_n$; he named it e because of its significance for the exponential function (or maybe after himself).

2. In the proof that a_n is increasing, the b could have been dispensed with, and replaced from the start with $1/2^{n+1}$. But this makes the proof harder to read, and obscures the simple algebra. Also, for greater clarity the proof is presented (as are many proofs) backwards from the natural procedure by which it would have been discovered; cf. Question 1.4/1.

3. In the proof that a_n is bounded by 3, it is easy enough to guess from the form of a_n that one should try the binomial theorem. Subsequent success then depends on a good estimation like (12), which shows the terms of the sum (10) are small. In general, this estimating lies at the very heart of analysis; it's an art which you learn by studying examples and working problems.

4. Notice how the three inequalities after line (10) as well as the two in line (13) are lined up one under the other. This makes the proof much easier to read and understand. When you write up your arguments, do the same thing: use separate lines and line up the $=$ and $\le$ symbols, so the proof can be read as successive transformations of the two sides of the equation or inequality.

Questions 1.4

1. Write down the proof that the sequence a_n is increasing as you think you would have discovered it. (In the Answers is one possibility, with a discussion of the problems of writing it up. Read it.)

2. Define $b_n = 1 + 1/1! + 1/2! + 1/3! + \ldots + 1/n!$; prove $\{b_n\}$ has a limit (it is e). (Hint: study the second half of the proof of Prop. 1.4.)

3. In the proof that $(1+1/k)^k$ is bounded above, the upper estimate 3 could be improved (i.e., lowered) by using more accurate estimates for the beginning terms of the sum on the right side of (10). If one only uses the estimate (11) when $i \geq 4$, what new upper bound does this give for $(1+1/k)^k$?

1.5 Example: the harmonic sum and Euler's number.

We consider an increasing sequence (its terms are called "harmonic sums") which does not have a limit. This somewhat subtle fact cannot even be guessed at by experimental calculation; it is only known because it can be proved. We will give two proofs for it.

Proposition 1.5A *Let* $a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\ , \quad n \geq 1\ .$

The sequence $\{a_n\}$ is strictly increasing, but not bounded above.

Proof 1. We will show that the terms $a_1, a_2, a_4, a_8, a_{16}, \ldots$ become arbitrarily large. This will show that $\{a_n\}$ is not bounded above.

Consider the term a_n, where $n = 2^k$. We write it out as follows, grouping the terms after the first two into groups of increasing length: $2, 4, 8, \ldots, 2^{k-1}$:

$$a_{2^k} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \ldots + \frac{1}{8}} + \underbrace{\frac{1}{9} + \ldots + \frac{1}{16}} + \ldots + \frac{1}{2^k}\ .$$

We have

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\ ,$$

$$\frac{1}{5} + \ldots + \frac{1}{8} > \frac{1}{8} + \ldots + \frac{1}{8} = \frac{1}{2}\ ,$$

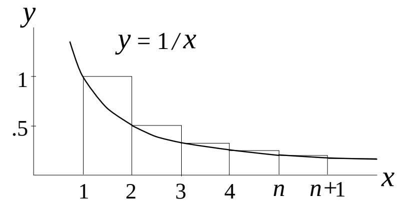
and so on. Thus each of the groupings has a sum $> 1/2$. Since there are $k-1$ such groupings, in addition to the beginning terms $1 + 1/2$, we get finally

$$a_{2^k} > 1 + \frac{1}{2} + (k-1)\left(\frac{1}{2}\right)\ ,$$

which shows that a_{2^k} becomes arbitrarily large as k increases. □

The next two proofs will use geometric facts about the graph of $1/x$ and the relation between areas and definite integrals. If you are after a completely logical, rigorous presentation of analysis, you can complain that these things haven't been defined yet. This is a valid objection, but we assume a reader who knows calculus already, wants to see how the ideas of analysis are used in familiar and unfamiliar settings, and is willing to wait for a rigorous presentation of the definite integral.

Proof 2. Draw the curve $y = 1/x$, and put in the rectangles shown, of width 1, and of height respectively $1,\ 1/2,\ 1/3, \ldots, 1/n$.



We compare the total area of the rectangles with the area under the curve between $x = 1$ and $x = n + 1$.

$$\text{total area of the rectangles} \;=\; 1 + \frac{1}{2} + \ldots + \frac{1}{n} \;=\; a_n \;;$$

$$\text{area under curve and over } [1, n+1] \;=\; \int_1^{n+1} \frac{dx}{x} \;=\; \ln(n+1)\ .$$

Since their tops lie above the curve, the rectangles have greater total area:

$$a_n \;>\; \ln(n+1)\ .$$

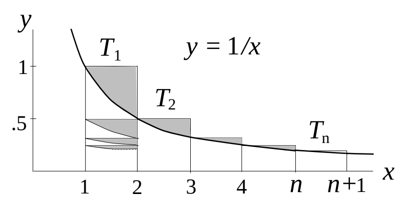
Since $\ln n$ increases without bound as n increases, so does a_n, and it follows that $\{a_n\}$ is not bounded above. □

Though this second proof is less elementary, it has the advantage of giving more insight into the approximate size of a_n than the first proof does. The picture suggests that a_n increases at about the same rate as $\ln(n+1)$. What can we say about the difference between them?

Proposition 1.5B *Let* $b_n = 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n} \;-\; \ln(n+1)\ , \quad n \geq 1.$

Then $\{b_n\}$ *has a limit (denoted by* γ *and called "Euler's number").*

Proof. It is sufficient to show $\{b_n\}$ is increasing and bounded above.



Referring to the picture at the right and the ideas of Proof 2 above, and letting T_i denote the area of the i-th shaded curvilinear triangle in the picture, we have

$$\begin{aligned} b_n &= \text{ area of rectangles } - \text{ area under curve} \\ &= T_1 + \ldots + T_n\ . \end{aligned}$$

The sequence $\{b_n\}$ is increasing, since $b_{n+1} = \; b_n + T_{n+1}$.

The sequence $\{b_n\}$ has 1 as an upper bound, since all of the "triangles" can be moved horizontally without overlapping into the rectangle of area 1 lying over the interval $1 \leq x \leq 2$. □

Remarks. How big is Euler's number $\gamma = \lim b_n$? From the proof,

$$\gamma \;=\; T_1 + T_2 + T_3 + \ldots \;=\; \text{total area of the triangles.}$$

Each of the triangles has a curved hypotenuse; if this were replaced by a straight side, the total area of the resulting triangles would add up to exactly half the area inside the rectangle over $[1, 2]$, i.e., to $1/2$. This shows that $1/2 < \gamma < 1$, but the picture suggests it is much closer to $1/2$.

It turns out that $\gamma = .577$, to three decimal places.

One of the mysteries about γ is whether or not it can be expressed in terms of other known numbers, like e, π, $\ln 2$ or $\sin 3$. It is also not known whether γ is an algebraic number, i.e., a zero of some polynomial with integer coefficients. These problems have been open for over 200 years.

We will meet γ again toward the end of this book.

Questions 1.5

1. Estimate the size of $1 + 1/2 + 1/3 + \ldots + 1/999$, by using the ideas of Proof 2. The approximation $\ln 10 \approx 2.3$ will be helpful (and is useful enough to be worth memorizing).

2. To see that Proposition 1.5A is not an experimental fact, suppose a computer adds up in a second 1,000,000 terms of the harmonic sum $\sum 1/k$. Estimating there are roughly 100,000,000 seconds/year, what will be the approximate value of a_n after one year of calculation? After two years of calculation?

3. In the proof of Proposition 1.5B, why is the phrase "without overlapping" included in the next-to-last line?

If the curve $1/x$ were replaced by the graph of some other function, what property should its graph have to guarantee a similar argument can be made?

1.6 Decreasing sequences. The Completeness Property.

To conclude, we gain a little more flexibility by first extending the notion of limit to decreasing sequences. The definition is the same as the one in Section 1.3; as before we assume that none of the a_n end with all 9's.

Definition 1.6A A number L, in a suitable decimal representation, is the **limit** of the decreasing sequence $\{a_n\}$ if, given any integer $k > 0$, all the a_n after some place in the sequence agree with L to k decimal places.

Definition 1.6B A sequence $\{a_n\}$ is said to be **bounded below** if there is a number C such that $a_n \geq C$ for all n.

Any such C is called a **lower bound** for the sequence.

Theorem 1.6 *A positive decreasing sequence has a limit.*

The plausibility argument is similar to the one we gave before and is omitted (think of the odometer on a car running in reverse). Note that since all the terms are positive, the sequence is bounded below by zero.

Theorems 1.3 and 1.6 can be extended to include sequences all or some of whose terms are not positive. Consider for example an increasing sequence $\{a_n\}$ which has a term ≤ 0. There are three cases:

a) The sequence also contains a positive term a_N. In this case, all the terms after a_N will be positive, and the argument for Theorem 1.3 applies.

b) All the terms are negative. In this case, just change the sign of all the terms: the sequence $\{-a_n\}$ will be a positive decreasing sequence, so it will have a limit L by Theorem 1.6; then $-L$ is the limit of $\{a_n\}$. For, since the decimal places of L agree with those of the $\{-a_n\}$, the places of $-L$ agree with those of the $\{a_n\}$.

c) Neither of the above. Left for you to figure out (see the Questions). $\square$

Decreasing sequences with a non-positive term can be handled similarly.

We would now like to combine all these cases into a single concise statement about the existence of a limit; it will be one of the cornerstones of our work in this book. For this we need two more words.

Definition 1.6C A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below; i.e., there are constants B and C such that

$$C \leq a_n \leq B \qquad \text{for all } n .$$

Notice that an increasing sequence is always bounded below (by its first term), so that for an increasing sequence it makes no difference whether we say it is bounded or bounded above. Similarly, saying a decreasing sequence is bounded below is the same as saying it is bounded.

Definition 1.6D A sequence is **monotone** if it is increasing for all n, or decreasing for all n.

> Humpty-Dumpty strikes again: in a rational world, "monotone" ought to be reserved for a sequence like $2, 2, 2, \ldots$, which is both increasing *and* decreasing. When you need a verbal macro, either you give a new meaning to an old word, or you coin a new one, like "scofflaw" for those who run speakeasies or red lights. Mathematicians do both.

We can summarize Theorems 1.3 and 1.6, allowing also non-positive terms, by the following statement; it is one form of what is called the *Completeness Property* of the real number system $\mathbb{R}$.

Completeness Property. *A bounded monotone sequence has a limit.*

The word "completeness" is used because the property says that the real line is "complete"— it has no holes. The early Greeks thought all numbers were rational; their line contained only points corresponding to the rational numbers. The discovery by Pythagoras that $\sqrt{2}$ is irrational (cf. Appendix A.2) was a mathematical earthquake; it meant that on the Greek line, there would be no limit for the sequence of points 1, 1.4, 1.41, 1.414, ..., since the line had no point representing the number $\sqrt{2}$. Passing from the rationals to the reals can be thought of as filling in the holes in the pre-Pythagorean line—making it complete, in other words.

Our definition of the limit of a sequence is reasonably intuitive, but has two defects. It works only for monotone sequences, and it is wedded too closely to decimal notation. We shall free it from both limitations in Chapter 3, but to do this, we need some ideas of estimation and approximation which are fundamental to all of analysis. So we turn to these in the next chapter.

Questions 1.6

1. For each of the a_n below, tell if the sequence $\{a_n\}$ is bounded or monotone; if both, give its limit. (Use $n = 0$ as the starting point, or $n = 1$ if a_0 would be undefined.)

(a) $1/n$ (b) $\sin 1/n$ (c) $\sin 4/n$ (d) $(-1)^n$ (e) $\ln(1/n)$

2. What is case (c) in the extension of the limit definition to increasing sequences with some non-positive terms? How would the argument for Theorem 1.3 go for this case?

Exercises (The exercises go with the indicated section of the chapter.)

1.2

1. For each of the a_n below, tell if the sequence $\{a_n\}$, $n \geq 1$, is increasing (strictly?), decreasing (strictly?), or neither; show reasoning.

(If simple inspection fails, try considering the difference $a_{n+1} - a_n$, or the ratio a_{n+1}/a_n, or relate the sequence to the values of a function $f(x)$ known to be increasing or decreasing.)

(a) $1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n}$ (b) $n/(n+1)$

(c) $\displaystyle\sum_1^n \sin^2 k$ (d) $\displaystyle\sum_1^n \sin k$

(e) $\tan(1/n)$ (f) $\sqrt{1 + 1/n^2}$

1.3

1. Show increasing; find an upper bound, if it exists; give the limit if you can.

(a) $\dfrac{\sqrt{n^2-1}}{n}$ (b) $\left(2-\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)$ (c) $\displaystyle\sum_0^n \sin^2 k\pi$ (d) $\displaystyle\sum_0^n \sin^2 k\pi/2$

2. Let $a_n = \sum_1^n 1/10^i$. Apply the method in Theorem 1.3 to find the limit L in its "suitable" decimal form; also express it as a rational number a/b.

3. Let $\{a_n\}$ be increasing, and $\lim\limits_{n\to\infty} a_n = L$, where L is a terminating decimal. Show that if $\{a_n\}$ is *strictly* increasing, the "suitable" decimal representation for L in Definition 1.3A is always the non-terminating form (ending with all 9's).

4. Read Section A.4 on mathematical induction (just the first page will be enough for now) and finish the argument for Theorem 1.3 by using induction.

1.4

1. Consider the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \frac{1}{1\cdot 3\cdot 5\cdot 7} + \ldots + \frac{1}{1\cdot 3\cdot \ldots \cdot (2n-1)} .$$

Decide whether $\{a_n\}$ is bounded above or not, and prove your answer is correct. (Hint: cf. Question 1.4/2 .)

2. Prove the sequence $a_n = n^n/n!$, $n \geq 1$, is

(a) increasing; (b) not bounded above (show $a_n \geq n$).

1.5

1. (a) Let $a_n = 1 + \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \ldots + \dfrac{1}{n(n+1)}$. Prove $\{a_n\}$ is bounded above. (Hint: $\dfrac{1}{2\cdot 3} = \dfrac{1}{2} - \dfrac{1}{3}$.)

(b) Let $b_n = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \ldots + \dfrac{1}{n^2}$. Prove $\{b_n\}$ is bounded above by comparing it to $\{a_n\}$. What upper bound does this give for $\{b_n\}$?

2. Prove the sequence $\{b_n\}$ of the preceding exercise is bounded, by expressing b_n as the area of a set of rectangles and comparing this with the area under a suitable curve. What upper bound does this give for $\{b_n\}$?

In fact, it is known that the limit of $\{b_n\}$ is $\pi^2/6$; how close is this to the bounds you got in this exercise and the preceding one?

3. Let $a_n = 1 + 1/\sqrt{2} + 1/\sqrt{3} + \ldots + 1/\sqrt{n}$. Prove $\{a_n\}$ is unbounded.

4. Let $b_n = a_n - 2\sqrt{n+1}$, where a_n is as in the previous exercise. Prove $\{b_n\}$ has a limit. (See Proposition 1.5B.)

1.6

1. Show the sequence $a_n = (n+1)/(n-1)$ is strictly decreasing and bounded below, and give its limit.

2. Show that $a_n = n/2^n$, $n \geq 1$, is a monotone sequence.

3. Define a sequence $\{a_n\}$ by: $a_{n+1} = 2{a_n}^2$; assume $0 < a_0 < 1/2$. Prove that a_n is strictly decreasing; is it bounded below?

4. Prove the sequence $a_n = \dfrac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}$ has a limit.

Problems

1-1 Define a sequence by

$$a_{n+1} = \frac{a_n + 1}{2}, \quad n \geq 0; \qquad a_0 \text{ arbitrary} .$$

(a) Prove that if $a_0 \leq 1$, the sequence is increasing and bounded above, and determine (without proof) its limit.

(b) Consider analogously the case $a_0 \geq 1$.

(c) Interpret the sequence geometrically as points on a line; this should make (a) and (b) intuitive.

1-2 Prove that $a_n = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right)$ is strictly increasing, and not bounded above.

1-3 Prove that $a_n = \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2 \cdot 4 \cdots (2n)}$ is strictly increasing and not bounded above.

1-4 Let A_n denote the area of the regular 2^n-sided polygon inscribed in a unit circle. (Assume $n \geq 2$.) Explain geometrically why the sequence $\{A_n\}$ is monotone and bounded above, and give its limit. Then use trigonometry to get an explicit expression for A_n, and prove the same facts analytically, using anything you know from calculus.

Answers to Questions

1.1

1. Example: $.099\ldots9 + .000\ldots1$ (both numbers have k decimal places). The sum sequence is $.0, .09, .099, \ldots, .099\ldots9$ ($k-1$ places), $.100\ldots0$ (k places), so the correct first decimal place of the sum—namely, 1—appears for the first time only in the k-th term of the sum sequence.

1.2

1. Only (b) is a sequence.

2. (a) $\{\sin n\pi/2\}$, $n \geq 0$ (b) $\left\{\dfrac{n}{n+1}\right\}$, $n \geq 1$, or $\left\{\dfrac{n+1}{n+2}\right\}$, $n \geq 0$.

3. (a) strictly decreasing (b) strictly increasing (c) strictly increasing (d) increasing (e) neither (f) strictly decreasing

1.3

1. (a) 1/2 or anything larger; (c) 1 or anything larger; (b) and (d) are not bounded above.

2. The limits are: (a) 1 (b) 1 (c) 2 (d) 2.

These or anything larger are upper bounds for the respective sequences.

3. The sequence is $.9, .99, .999, \ldots$; following the method in Theorem 1.3 gives the limit $L = .999\ldots$, i.e., the non-terminating form of 1.

4. The argument says at various points: the integer parts increase; after a certain point, the first decimal place increases; after a later point, the second decimal place increases, and so on.

This would not necessarily be true if we did not require a uniform choice (terminating form) for the decimal expansions of the a_n. For instance, the constant sequence $1.000, .999\ldots, 1.000, \ldots$, whose representation alternates between the terminating and non-terminating forms of 1, is increasing and bounded, yet the integer part of its terms is not increasing.

Similarly, in the constant sequence $1.300, 1.299\ldots, 1.300$, the integer part is unchanging, but the sequence formed by the first decimal places $3, 2, 3, \ldots$ is not an increasing sequence.

1.4

1. The following is a fairly common method, both of discovering the argument that a_n is increasing, and writing it up.

$$\begin{aligned} a_{n+1} &\geq a_n \; ; && (1)\\ \left(1+\frac{1}{2^{n+1}}\right)^{2^{n+1}} &\geq \left(1+\frac{1}{2^n}\right)^{2^n} \; ; && (2)\\ \left(1+\frac{1}{2^{n+1}}\right)^{2} &\geq \left(1+\frac{1}{2^n}\right) \; ; && (3)\\ 1+\frac{2}{2^{n+1}}+\frac{1}{2^{2(n+1)}} &\geq 1+\frac{1}{2^n} \; ; && (4)\\ \frac{1}{2^{2(n+1)}} &\geq 0 \; . && (5) \end{aligned}$$

At this point (or at step (4)), the problem is considered solved.

The meaning of this falling-domino argument is presumably:

(1) is true if (2) is true: (2) $\Rightarrow$ (1);
(2) is true if (3) is true: (3) $\Rightarrow$ (2);
(3) is true if (4) is true: (4) $\Rightarrow$ (3);
(4) is true if (5) is true: (5) $\Rightarrow$ (4);
(5) is true. Therefore (1) is true.

The argument is written backwards. Students often try to express this by writing the first four inequalities using an invented symbol, such as $\geq ?$. A formal argument would usually write the successive inequalities in the opposite order, without any ? symbols. But written in this way, the argument is unmotivated, and can be difficult to follow, since the readers can't tell where they are headed.

A reasonable compromise might be to use some symbol like $\geq ?$, with the understanding that the next line then *must* imply the line containing the $\geq ?$.

2. By Theorem 1.3 it suffices to show that $\{b_n\}$ is increasing and bounded above. (The limit is again e.)

It is strictly increasing since $b_{n+1} = b_n + 1/(n+1)!$.

It is bounded above since

$$\begin{aligned} b_n &= 1 + 1/1! + 1/2! + 1/3! + \dots \\ &\le 1 + 1 + 1/2 + 1/2^2 + 1/2^3 + \dots, \qquad \text{by (11);} \\ &= 3 . \end{aligned}$$

3. Using the estimate (11) only starting with the fifth term, we have

$$\begin{aligned} \left(1+\frac{1}{k}\right)^k &= 1 + k\left(\frac{1}{k}\right) + \dots + \frac{k(k-1)\cdots(k-i+1)}{i!}\left(\frac{1}{k}\right)^i + \dots ; \\ &\le\ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{2^3} + \frac{1}{2^4} + \dots ; \\ &\le \frac{8}{3} + \frac{1}{2^3}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right); \\ &\le 2.67 + .25 = 2.92 . \end{aligned}$$

1.5

1. $0 < a_{999} - \ln(1000) < 1$ by proofs 1.5A[2], 1.5B; $\ln(1000) \approx 6.9$.

2. $\ln(10^{14}) \approx 14(2.3) \approx 32.2$, after one year;

$\ln(2 \cdot 10^{14}) = \ln 2 + \ln 10^{14} \approx .7 + 32.2 \approx 32.9$, after two years.

3. The statement (total area of triangles) $\le$ (area of rectangle) can only be made if the triangles do not overlap when fitted inside the rectangle.

If the function is positive and strictly increasing or strictly decreasing, the triangles will not overlap.

1.6

1. (a) both (decreasing); 0 (b) both (decreasing); 0

(c) bounded, not monotone (d) bounded, not monotone

(e) monotone (decreasing), not bounded

2. Case (c): The increasing sequence contains no positive terms, but not all the terms are negative.

The sequence then contains the term 0, but no positive terms. Since it is increasing, all terms after 0 must also be 0, so the limit exists and is 0.