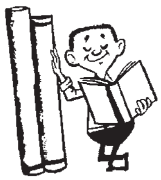
Hypergeometric Functions



Reading	Problems

Introduction

The hypergeometric function $F(a,\ b;\ c;\ x)$ is defined as

$$\begin{aligned}
F(a;\ b;\ c;\ x) &= {}_2F_1(a,\ b;\ c;\ x) = F(b,\ a;\ c;\ x) \\
&= 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!} \qquad |x| < 1, \quad c \neq 0, -1, -2, \ldots
\end{aligned}$$

where

2 – refers to number of parameters in numerator
1 – refers to number of parameters in denominator

$$y_1 = F(a,\ b;\ c;\ x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n))\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}$$

is one solution of the Hypergeometric Differential Equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

The other independent solution is

$$y_2 = x^{1-c}F(a-c+1,\ b-c+1;\ 2-c;\ x) \qquad c \neq -1, -2, \ldots$$

Some Properties of $F(a,\ b;\ c;\ x)$

$$\frac{d}{dx}F(a,\ b;\ c;\ x) = \frac{ab}{c}F(a+1, b+1;\ c+1;\ x)$$

The general formula based on repeated differentiation

$$\frac{d^k}{dx^k}\ F(a,\ b;\ c;\ x) = \frac{(a)_k(b)_k}{(c)_k}\ F(a+k), b+k;\ c+k; x), \qquad k = 1, 2, 3, \ldots$$

Recurrence Relations

There are 15 recurrence relations, one of the simplest is

$$(a-b)F(a,\ b;\ c; x) = aF(a+1,\ b;\ c;\ x) - bF(a,\ b+1;\ c; x)$$

See Abramowitz and Stegun for others.

Integral Representation

$$F(a,\ b;\ c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\ dt \qquad |x| \leq 1$$

Generating Function

$$w(x,t) = (1-t)^{b-c}(1-t+xt)^{-b}, \qquad c \neq 0, -1, -2, \ldots$$

$$w(x,t) = \sum_{n=0}^{\infty}\frac{(c)_n}{n!}\ F(-n, b;\ c;\ x)t^n$$

where $\boldsymbol{F(-n,\ b;\ c;\ x)}$ denotes hypergeometric polynomials

$$F(-n,\ b;\ c;\ x) = \sum_{i=0}^{n} \frac{(-n)\ (b)}{(c)} \frac{x^i}{i!} \qquad\qquad -\infty < x < \infty$$

Relation to Other Functions

Beta Functions

$$B_x(p,q) \;=\; \frac{x^p}{p}\ F(p,\ 1-q;\ 1+p;\ x)$$

$$B_1(p,q) \;=\; \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Elliptic Integrals

$$\mathrm{K}(k) = \frac{\pi}{2}\ F\left(\frac{1}{2}, \frac{1}{2};\ 1;\ k^2\right)$$

$$\mathrm{E}(k) = \frac{\pi}{2}\ F\left(-\frac{1}{2}, \frac{1}{2};\ 1;\ k^2\right)$$

Legendre Functions

$$\mathrm{P}_\nu(x) = F\left(-\nu,\ \nu+1;\ 1;\ \frac{1-x}{2}\right) \qquad\qquad \nu \neq integer$$

$$\mathrm{Q}_\nu(x) = \frac{\pi^{1/2}}{2^{\nu+1}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)} \frac{1}{x^{\nu+1}}\ F\left(1+\frac{\nu}{2}, \frac{1+\nu}{2};\ \nu+\frac{3}{2};\ \frac{1}{x^2}\right) \qquad |x| > 1$$

Chebyshev Polynomials

$$T_n(x) = F\left(-n,\ n;\ \frac{1}{2};\ \frac{1-x}{2}\right)$$

$$U_n(x) = (n+1)F\left(-n,\ n+2;\ \frac{3}{2};\ \frac{1-x}{2}\right)$$

Bessel Function

$$J_\nu = \frac{(x/2)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(\nu+1; -\frac{x^2}{4}\right)$$

Confluent Hypergeometric Functions (Kummer's Function)

Functions $M(a;\ c;\ x)$ and $U(a;\ c;\ x)$

$$M(a;\ c;\ x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!} \qquad -\infty < x < \infty$$

$$M(a;\ c;\ x) = \lim_{b\to\infty} F(a,\ b;\ c;\ x/b) \qquad c \neq 0, -1, -2 \ldots$$

also written as

$${}_1F_1(a;\ c;\ x)$$

Differential Equation

$$xy'' + (c-x)y' - ay = 0$$

If we change the variable to $y = x^{1-c}z$, then

$$xz'' + (2-c-x)z' - (1+a-c)z = 0$$

where the solution is

$$z = M(1+a-c;\ 2-c;\ x) \qquad c \neq 2, 3, 4, \ldots$$

and

$$y_2 = x^{1-c}M(1+a-c;\ 2-c;\ x) \qquad c \neq 2, 3, 4, \ldots$$

is a second solution.

Therefore the complete solution is

$$\begin{aligned} y &= y_1 + y_2 \\ &= C_1 M(a;\ c;\ x) + C_2 M(1+a-c;\ 2-c;\ x) \qquad c \neq 0, \pm 1, \pm 2 \ldots \end{aligned}$$

Confluent Hypergeometric Function of the Second Kind

$$U(a;\ c;\ x) = \frac{\pi}{\sin c\pi}\left[\frac{M(a;\ c;\ x)}{\Gamma(1+a-c)\Gamma(c)} - \frac{x^{1-c}M(1+a-c;\ 2-c;\ x)}{\Gamma(a)\Gamma(2-c)}\right]$$

$$c \neq 0, -1, -2. \ldots$$

Integral Representations

$$M(a;\ c;\ x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1}\, dt \qquad c > a > 0$$

$$U(a;\ c;\ x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1}\, dt \qquad a > 0,\ x > 0$$

with $t = 1 - u$

$$M(a;\ c;\ x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\ e^x \int_0^1 e^{-xu} u^{c-a-1} (1-u)^{a-1}\, du$$

Therefore

$$M(a;\ c;\ x) = e^x M(c-a;\ c;\ -x) \qquad \text{Kummer's Transformation}$$

for many other properties see Abramowitz and Stegun

Relation to Other Functions

$$e^x = M(a;\ a;\ x)$$

$$\text{erfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}\ U\left(\frac{1}{2};\ \frac{1}{2};\ x^2\right)$$

$$\gamma(a, x) = \frac{x^a}{a}\ M(a;\ a+1;\ -x)$$

$$\Gamma(a, x) = e^{-x}\ U(1-a;\ 1-a;\ x)$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} x\ M\left(\frac{1}{2};\ \frac{3}{2};\ -x^2\right)$$

$$J_p(x) = \frac{(x/2)^p}{\Gamma(p+1)} e^{-ix}\ M\left(p + \frac{1}{2};\ 2p+1;\ 2ix\right)$$

$$I_p(x) = \frac{(x/2)^p}{\Gamma(p+1)} e^{-x}\ M\left(p + \frac{1}{2};\ 2p+1;\ 2x\right)$$

$$K_p(x) = \sqrt{\pi} (2x)^p e^{-x}\ U\left(p + \frac{1}{2};\ 2p+1;\ 2x\right)$$