1.2 Elementary Row Operations

Example 1.2.1 *Find all solutions of the following system :*

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$$

In other (perhaps simpler) examples we were able to find solutions by simplifying the system (perhaps by eliminating certain variables) through operations of the following types :

1. We could multiply one equation by a non-zero constant.
2. We could add one equation to another (for example in the hope of eliminating a variable from the result).

A similar approach will work for Example 1.2.1 but with this and other harder examples it may not always be clear how to proceed. We now develop a new technique both for describing our system and for applying operations of the above types more systematically and with greater clarity.

Back to Example 1.2.1: We associate a *matrix* to our system of equations (a matrix is a rectangular array of numbers).

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix} \begin{array}{l} \text{Eqn } 1 \\ \text{Eqn } 2 \\ \text{Eqn } 3 \end{array}$$

Note that the first *row* of this matrix contains as its four entries the coefficients of the variables x, y, z, and the number appearing on the right-hand-side of Equation 1 of the system. Rows 2 and 3 correspond similarly to Equations 2 and 3. The *columns* of the matrix correspond (from left to right) to the variables $x,\ y,\ z$ and the right hand side of our system of equations.

Definition 1.2.2 *The above matrix is called the* augmented matrix *of the system of equations in Example 1.2.1.*

In solving systems of equations we are allowed to perform operations of the following types:

1. Multiply an equation by a non-zero constant.
2. Add one equation (or a non-zero constant multiple of one equation) to another equation.

These correspond to the following operations on the augmented matrix :

1. Multiply a *row* by a non-zero constant.
2. Add a multiple of one row to another row.
3. We also allow operations of the following type : Interchange two rows in the matrix (this only amounts to writing down the equations of the system in a different order).

Definition 1.2.3 *Operations of these three types are called* Elementary Row Operations (ERO's) *on a matrix.*

We now describe how ERO's on the augmented matrix can be used to solve the system of Example 1.2.1. The following table describes how an ERO is performed at each step to produce a new augmented matrix corresponding to a new (hopefully simpler) system.

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$
1. $R3 \to R3 + R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ & & 6y & + & z & = & 5 \end{array}$
2. $R2 \to R2 - 3R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & - & 5y & + & z & = & -6 \\ & & 6y & + & z & = & 5 \end{array}$
3. $R2 \to R2 + R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$
4. $R3 \to R3 - 6R2$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -11 & 11 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & -11z & = & 11 \end{array}$
5. $R3 \times \left(-\frac{1}{11}\right)$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcrl} x & + & 2y & - & z & = & 5 & (A) \\ & & y & + & 2z & = & -1 & (B) \\ & & & & z & = & -1 & (C) \end{array}$

We have produced a new system of equations. This is easily solved :

<u>Backsubstitution</u>
$$\begin{cases} (C) & z = -1 \\ (B) & y = -1 - 2z \quad \Longrightarrow \quad y = -1 - 2(-1) = 1 \\ (A) & x = 5 - 2y + z \quad \Longrightarrow \quad x = 5 - 2(1) + (-1) = 2 \end{cases}$$

Solution : $\boxed{x = 2,\ y = 1, z = -1}$

You should check that this is a solution of the original system. It is the only solution both of the final system and of the original one (and every intermediate one).

<u>Note</u> : The matrix obtained in Step 5 above is in *Row-Echelon Form*. This means :

1. The first non-zero entry in each row is a 1 (called a *Leading 1*).

2. If a column contains a leading 1, then every entry of the column below the leading 1 is a zero.

3. As we move downwards through the rows of the matrix, the leading 1's move from left to right.

4. Any rows consisting entirely of zeroes are grouped together at the bottom of the matrix.

Note : The process by which the augmented matrix of a system of equations is reduced to row-echelon form is called *Gaussian Elimination*. In Example 1.2.1 the solution of the system was found by Gaussian elimination with *Backsubstitution*.

General Strategy to Obtain a Row-Echelon Form

1. Get a 1 as the top left entry of the matrix.

2. Use this first leading 1 to "clear out" the rest of the first column, by adding suitable multiples of Row 1 to subsequent rows.

3. If column 2 contains non-zero entries (other than in the first row), use ERO's to get a 1 as the second entry of Row 2. After this step the matrix will look like the following (where the entries represented by stars may be anything):

$$\begin{pmatrix} 1 & * & * & \dots & \dots \\ 0 & 1 & \dots & \dots & \dots \\ 0 & * & \dots & \dots & \dots \\ 0 & * & \dots & \dots & \dots \\ \vdots & \vdots & & & \vdots \\ 0 & * & \dots & \dots & \dots \end{pmatrix}$$

4. Now use this second leading 1 to "clear out" the rest of column 2 (below Row 2) by adding suitable multiples of Row 2 to subsequent rows. After this step the matrix will look like

the following :

$$\begin{pmatrix} 1 & * & * & \dots & \dots \\ 0 & 1 & * & \dots & \dots \\ 0 & 0 & * & \dots & \dots \\ 0 & 0 & * & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \dots & \dots \end{pmatrix}$$

5. Now go to column 3. If it has non-zero entries (other than in the first two rows) get a 1 as the third entry of Row 3. Use this third leading 1 to clear out the rest of Column 3, then proceed to column 4. Continue until a row-echelon form is obtained.

Example 1.2.4 *Let A be the matrix*

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$$

Reduce A to row-echelon form.

Solution:

1. Get a 1 as the first entry of Row 1. Done.

2. Use this first leading 1 to clear out column 1 as follows :

$$\begin{array}{lcl} R2 & \to & R2 - 2R1 \\ R3 & \to & R3 - R1 \end{array} \quad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 3 & 1 & -2 & 8 \\ 0 & -2 & 3 & 5 & 2 \end{pmatrix}$$

3. Get a leading 1 as the second entry of Row 2, for example as follows :

$$R2 \to R2 + R3 \quad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & -2 & 3 & 5 & 2 \end{pmatrix}$$

4. Use this leading 1 to clear out whatever appears below it in Column 2 :

$$R3 \to R3 + 2R2 \quad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 11 & 11 & 22 \end{pmatrix}$$

5. Get a leading 1 in Row 3 :

$$R3 \quad \times \quad \frac{1}{11} \qquad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

This matrix is now in row-echelon form.

Definition 1.2.5 *Let A be a matrix. The* rank *of A, denoted rank(A) is the number of leading 1's in a row-echelon form obtained from A by Gaussian elimination as above.*

Remarks :

1. The rank of the matrix A in Example 1.2.4 is 3, since the row-echelon form obtained had 3 leading 1's (one in each row).

2. The rank of any matrix can be at most equal to the number of rows, since each row in a REF (row-echelon form) can contain at most one leading 1. If a REF obtained from some matrix contains rows full of zeroes, the rank of this matrix will be less than the number of rows.

3. Starting with a particular matrix, different sequences of ERO's can lead to different row-echelon forms. However, all have the same rank.