Bessel functions

The Bessel function $J_\nu(z)$ of the first kind of order ν is defined by

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix};-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!}\left(\frac{z}{2}\right)^{2k}. \tag{1}$$

For $\nu \geq 0$ this is a solution of the Bessel differential equation

$$z^2 y''(z) + zy'(z) + \left(z^2 - \nu^2\right) y(z) = 0, \quad \nu \geq 0. \tag{2}$$

For $\nu \notin \{0,1,2,\ldots\}$ we have that $J_{-\nu}(z)$ is a second solution of the differential equation (2) and the two solutions $J_\nu(z)$ and $J_{-\nu}(z)$ are clearly linearly independent.
For $\nu = n \in \{0,1,2,\ldots\}$ we have

$$J_{-n}(z) = \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(-n+k+1)\,k!}\left(\frac{z}{2}\right)^{2k} = \left(\frac{z}{2}\right)^{-n} \sum_{k=n}^\infty \frac{(-1)^k}{\Gamma(-n+k+1)\,k!}\left(\frac{z}{2}\right)^{2k},$$

since

$$\frac{1}{\Gamma(-n+k+1)} = 0 \quad \text{for} \quad k = 0,1,2,\ldots,n-1.$$

This implies that

$$\begin{aligned} J_{-n}(z) &= \left(\frac{z}{2}\right)^{-n} \sum_{k=n}^\infty \frac{(-1)^k}{\Gamma(-n+k+1)\,k!}\left(\frac{z}{2}\right)^{2k} = \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^\infty \frac{(-1)^{n+k}}{\Gamma(k+1)\,(n+k)!}\left(\frac{z}{2}\right)^{2(n+k)} \\ &= (-1)^n \left(\frac{z}{2}\right)^{n} \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(n+k+1)\,k!}\left(\frac{z}{2}\right)^{2k} = (-1)^n J_n(z). \end{aligned}$$

This implies that $J_n(z)$ and $J_{-n}(z)$ are linearly dependent for $n \in \{0,1,2,\ldots\}$.
A second linearly independent solution can be found as follows. Since $(-1)^n = \cos n\pi$, we see that $J_\nu(z)\cos\nu\pi - J_{-\nu}(z)$ is a solution of the differential equation (2) which vanishes when $\nu = n \in \{0,1,2,\ldots\}$. Now we define

$$Y_\nu(z) := \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \tag{3}$$

where the case that $\nu = n \in \{0,1,2,\ldots\}$ should be regarded as a limit case. By l'Hopital's rule we have

$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial\nu}\right]_{\nu=n} - \frac{(-1)^n}{\pi}\left[\frac{\partial J_{-\nu}(z)}{\partial\nu}\right]_{\nu=n}.$$

This implies that $Y_{-n}(z) = (-1)^n Y_n(z)$ for $n \in \{0,1,2,\ldots\}$.
The function $Y_\nu(z)$ is called the Bessel function of the second kind of order ν.
Using the definition (1) we find that

$$\left[\frac{\partial J_\nu(z)}{\partial\nu}\right]_{\nu=n} = J_n(z)\ln\left(\frac{z}{2}\right) - \left(\frac{z}{2}\right)^n \sum_{k=0}^\infty \frac{(-1)^k\psi(n+k+1)}{(n+k)!\,k!}\left(\frac{z}{2}\right)^{2k},$$

where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

For $\nu \notin \{0, 1, 2, \ldots\}$ we have

$$\frac{\partial J_{-\nu}(z)}{\partial \nu} = -J_{\nu}(z) \ln \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(-\nu + k + 1)}{\Gamma(-\nu + k + 1)\, k!} \left(\frac{z}{2}\right)^{2k}.$$

Now we use

$$\lim_{z \to -n} (z + n)\Gamma(z) = \frac{(-1)^n}{n!} \quad \Longrightarrow \quad \Gamma(z) \sim \frac{(-1)^n}{(z + n)\, n!} \quad \text{for} \quad z \to -n$$

and

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \sim \frac{-\frac{(-1)^n}{(z+n)^2\, n!}}{\frac{(-1)^n}{(z+n)\, n!}} = -\frac{1}{z + n} \quad \text{for} \quad z \to -n.$$

This implies that

$$\lim_{z \to -n} \frac{\psi(z)}{\Gamma(z)} = (-1)^{n+1} n! \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Hence

$$\begin{aligned}
&\lim_{\nu \to n} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(-\nu + k + 1)}{\Gamma(-\nu + k + 1)\, k!} \left(\frac{z}{2}\right)^{2k} \\
&\quad = (-1)^n \sum_{k=0}^{n-1} \frac{(n - k - 1)!}{k!} \left(\frac{z}{2}\right)^{2k} + \sum_{k=n}^{\infty} \frac{(-1)^k \psi(-n + k + 1)}{\Gamma(-n + k + 1)\, k!} \left(\frac{z}{2}\right)^{2k} \\
&\quad = (-1)^n \sum_{k=0}^{n-1} \frac{(n - k - 1)!}{k!} \left(\frac{z}{2}\right)^{2k} + (-1)^n \left(\frac{z}{2}\right)^{2n} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k + 1)}{\Gamma(k + 1)\, (n + k)!} \left(\frac{z}{2}\right)^{2k}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\left[\frac{\partial J_{-\nu}(z)}{\partial \nu}\right]_{\nu = n} &= -J_{-n}(z) \ln \left(\frac{z}{2}\right) + (-1)^n \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n - k - 1)!}{k!} \left(\frac{z}{2}\right)^{2k} \\
&\quad + (-1)^n \left(\frac{z}{2}\right)^{n} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k + 1)}{(n + k)!\, k!} \left(\frac{z}{2}\right)^{2k}.
\end{aligned}$$

Finally we use the fact that $J_{-n}(z) = (-1)^n J_n(z)$ to conclude that

$$\begin{aligned}
Y_n(z) &= \frac{2}{\pi} J_n(z) \ln \left(\frac{z}{2}\right) - \frac{1}{\pi} \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n - k - 1)!}{k!} \left(\frac{z}{2}\right)^{2k} \\
&\quad - \frac{1}{\pi} \left(\frac{z}{2}\right)^{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n + k)!\, k!} \left[\psi(n + k + 1) + \psi(k + 1)\right] \left(\frac{z}{2}\right)^{2k}
\end{aligned}$$

for $n \in \{0, 1, 2, \ldots\}$. Compare with the theory of Frobenius for linear second differential equations.

In the theory of second order linear differential equations of the form

$$y'' + p(z)y' + q(z)y = 0,$$

two solutions y_1 and y_2 are linearly independent if and only if

$$W(y_1, y_2)(z) := \begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{vmatrix} = y_1(z)y_2'(z) - y_1'(z)y_2(z) \neq 0.$$

This determinant is called the Wronskian of the solutions y_1 and y_2.
It can (easily) be shown that this determinant of Wronski satisfies the differential equation

$$W'(z) + p(z)W(z) = 0.$$

This result is called Abel's theorem or the theorem of Abel-Liouville. In the case of the Bessel differential equation we have $p(z) = 1/z$, which implies that

$$W'(z) + \frac{1}{z}W(z) = 0 \quad \Longrightarrow \quad W(y_1, y_2)(z) = \frac{c}{z}$$

for some constant c. Now we have

Theorem 1.

$$W(J_\nu, J_{-\nu})(z) = -\frac{2\sin\nu\pi}{\pi z} \quad \textit{and} \quad W(J_\nu, Y_\nu)(z) = \frac{2}{\pi z}. \tag{4}$$

For $\nu \geq 0$ this implies that $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent if $\nu \notin \{0, 1, 2, \ldots\}$ and that $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all $\nu \geq 0$.

Proof. Note that

$$\begin{aligned} W(J_\nu, Y_\nu)(z) &= J_\nu(z)Y_\nu'(z) - J_\nu'(z)Y_\nu(z) \\ &= J_\nu(z)\frac{J_\nu'(z)\cos\nu\pi - J_{-\nu}'(z)}{\sin\nu\pi} - J_\nu'(z)\frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \\ &= -\frac{J_\nu(z)J_{-\nu}'(z) - J_\nu'(z)J_{-\nu}(z)}{\sin\nu\pi} = -\frac{W(J_\nu, J_{-\nu})(z)}{\sin\nu\pi}. \end{aligned}$$

Now we use the definition (1) to obtain

$$J_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!} \cdot \frac{z^{\nu+2k}}{2^{\nu+2k}} \quad \Longrightarrow \quad J_\nu'(z) = \sum_{k=0}^\infty \frac{(-1)^k(\nu+2k)}{\Gamma(\nu+k+1)\,k!} \cdot \frac{z^{\nu+2k-1}}{2^{\nu+2k}}$$

and

$$J_{-\nu}(z) = \sum_{m=0}^\infty \frac{(-1)^m}{\Gamma(-\nu+m+1)\,m!} \cdot \frac{z^{-\nu+2m}}{2^{-\nu+2m}}$$

$$\Longrightarrow \quad J_{-\nu}'(z) = \sum_{m=0}^\infty \frac{(-1)^m(-\nu+2m)}{\Gamma(-\nu+m+1)\,m!} \cdot \frac{z^{-\nu+2m-1}}{2^{-\nu+2m}}.$$

Hence we have

$$\begin{aligned} z\,W(J_\nu, J_{-\nu})(z) &= z\left[J_\nu(z)J'_{-\nu}(z) - J'_\nu(z)J_{-\nu}(z)\right] \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}(-\nu+2m)}{\Gamma(\nu+k+1)\Gamma(-\nu+m+1)\,k!\,m!}\cdot\frac{z^{2k+2m}}{2^{2k+2m}} \\ &\qquad - \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}(\nu+2k)}{\Gamma(\nu+k+1)\Gamma(-\nu+m+1)\,k!\,m!}\cdot\frac{z^{2k+2m}}{2^{2k+2m}} \\ &= -\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}(2\nu+2k-2m)}{\Gamma(\nu+k+1)\Gamma(-\nu+m+1)\,k!\,m!}\cdot\frac{z^{2k+2m}}{2^{2k+2m}}. \end{aligned}$$

This implies that

$$\lim_{z\to 0} z\,W(J_\nu, J_{-\nu})(z) = -\frac{2\nu}{\Gamma(\nu+1)\Gamma(-\nu+1)} = -\frac{2\nu}{\nu\Gamma(\nu)\Gamma(1-\nu)} = -\frac{2\sin\nu\pi}{\pi}.$$

Using the definition (1) we obtain

$$\begin{aligned} \frac{d}{dz}\left[z^\nu J_\nu(z)\right] &= \frac{d}{dz}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)\,k!}\cdot\frac{z^{2\nu+2k}}{2^{\nu+2k}} = \sum_{k=0}^{\infty}\frac{(-1)^k(2\nu+2k)}{\Gamma(\nu+k+1)\,k!}\cdot\frac{z^{2\nu+2k-1}}{2^{\nu+2k}} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k)\,k!}\cdot\frac{z^{2\nu+2k-1}}{2^{\nu+2k-1}} = z^\nu J_{\nu-1}(z). \end{aligned}$$

Hence we have

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right] = z^\nu J_{\nu-1}(z) \quad\Longleftrightarrow\quad zJ'_\nu(z) + \nu J_\nu(z) = zJ_{\nu-1}(z). \tag{5}$$

Similarly we have

$$\begin{aligned} \frac{d}{dz}\left[z^{-\nu} J_\nu(z)\right] &= \frac{d}{dz}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)\,k!}\cdot\frac{z^{2k}}{2^{\nu+2k}} = \sum_{k=0}^{\infty}\frac{(-1)^k\,2k}{\Gamma(\nu+k+1)\,k!}\cdot\frac{z^{2k-1}}{2^{\nu+2k}} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\Gamma(\nu+k+2)\,k!}\cdot\frac{z^{2k+1}}{2^{\nu+2k+1}} = -z^{-\nu}J_{\nu+1}(z). \end{aligned}$$

Hence we have

$$\frac{d}{dz}\left[z^{-\nu} J_\nu(z)\right] = -z^{-\nu} J_{\nu+1}(z) \quad\Longleftrightarrow\quad zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z). \tag{6}$$

Elimination of $J'_\nu(z)$ from (5) and (6) gives

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z)$$

and elimination of $J_\nu(z)$ from (5) and (6) gives

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z).$$

Special cases

For $\nu = 1/2$ we have from the definition (1) by using Legendre's duplication formula for the gamma function

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\,k!} \left(\frac{x}{2}\right)^{2k} = \sqrt{\frac{2x}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} = \sqrt{\frac{2}{\pi x}} \sin x, \quad x > 0$$

and for $\nu = -1/2$ we have

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1/2)\,k!} \left(\frac{x}{2}\right)^{2k} = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \sqrt{\frac{2}{\pi x}} \cos x, \quad x > 0.$$

Note that the definition (3) implies that

$$Y_{1/2}(x) = -J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x \quad \text{and} \quad Y_{-1/2}(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad x > 0.$$

Integral representations

First we will prove

Theorem 2.

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}} \left(\frac{z}{2}\right)^{\nu} \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-1/2}\,dt, \quad \operatorname{Re}\nu > -1/2. \tag{7}$$

Proof. We start with

$$\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-1/2}\,dt = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \int_{-1}^{1} t^n (1-t^2)^{\nu-1/2}\,dt.$$

Note that the latter integral vanishes when n is odd. For $n = 2k$ we obtain using $t^2 = u$

$$\begin{aligned}\int_{-1}^{1} t^{2k}(1-t^2)^{\nu-1/2}\,dt &= 2\int_0^1 u^k(1-u)^{\nu-1/2}\,\frac{du}{2\sqrt{u}} = \int_0^1 u^{k-1/2}(1-u)^{\nu-1/2}\,du \\ &= B(k+1/2, \nu+1/2) = \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(\nu+k+1)}.\end{aligned}$$

Now we use Legendre's duplication formula to find that

$$\Gamma(k+1/2) = \frac{\sqrt{\pi}}{2^{2k-1}} \cdot \frac{\Gamma(2k)}{\Gamma(k)} = \frac{\sqrt{\pi}}{2^{2k}} \cdot \frac{\Gamma(2k+1)}{\Gamma(k+1)} = \frac{\sqrt{\pi}}{2^{2k}} \cdot \frac{(2k)!}{k!}.$$

Hence we have

$$\begin{aligned}\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-1/2}\,dt &= \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} \cdot \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(\nu+k+1)} \\ &= \Gamma(\nu+1/2)\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!} \left(\frac{z}{2}\right)^{2k}.\end{aligned}$$

This proves the theorem.

We also have Poisson's integral representations

Theorem 3.

$$
\begin{aligned}
J_\nu(z) &= \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_0^\pi e^{iz\cos\theta}\,(\sin\theta)^{2\nu}\,d\theta \\
&= \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_0^\pi \cos(z\cos\theta)\,(\sin\theta)^{2\nu}\,d\theta, \quad \operatorname{Re}\nu > -1/2.
\end{aligned}
$$

Proof. Use the substitution $t = \cos\theta$ to obtain

$$
\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-1/2}\,dt = \int_\pi^0 e^{iz\cos\theta}(1-\cos^2\theta)^{\nu-1/2}\cdot(-\sin\theta)\,d\theta = \int_0^\pi e^{iz\cos\theta}(\sin\theta)^{2\nu}\,d\theta.
$$

Further we have

$$
e^{iz\cos\theta} = \cos(z\cos\theta) + i\sin(z\cos\theta)
$$

and

$$
\int_0^\pi \sin(z\cos\theta)(\sin\theta)^{2\nu}\,d\theta = 0.
$$

This shows that Poisson's integral representations follow from the integral representation (7).

Remarks:

1. The Fourier transform is defined by

$$
F(z) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-izt}f(t)\,dt
$$

with inversion formula

$$
f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{izt}F(z)\,dz.
$$

This implies that the Fourier transform of the function

$$
f(t) = \begin{cases} (1-t^2)^{\nu-1/2}, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}
$$

is

$$
F(z) = \frac{\Gamma(\nu+1/2)}{\sqrt{2}}\left(\frac{z}{2}\right)^{-\nu} J_\nu(z).
$$

2. Instead of the substitution $t = \cos\theta$ in (7) one can also use the substitution $t = \sin\theta$, which leads to slightly different forms of Poisson's integral representations. In fact we have

$$
\begin{aligned}
J_\nu(z) &= \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_{-\pi/2}^{\pi/2} e^{iz\sin\theta}\,(\cos\theta)^{2\nu}\,d\theta \\
&= \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_{-\pi/2}^{\pi/2} \cos(z\sin\theta)\,(\cos\theta)^{2\nu}\,d\theta \\
&= \frac{2}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \cos(z\sin\theta)\,(\cos\theta)^{2\nu}\,d\theta, \quad \operatorname{Re}\nu > -1/2.
\end{aligned}
$$

Integrals of Bessel functions

The Hankel transform of a function f is defined by

$$F(s)=\int_0^\infty tf(t)J_\nu(st)\,dt$$

for functions f for which the integral converges. The inversion formula is given by

$$f(t)=\int_0^\infty sF(s)J_\nu(st)\,ds.$$

This pair of integrals is called a Hankel pair of order ν.

An example of such an integral is

$$\int_0^\infty t^{\mu-1}e^{-\rho^2t^2}J_\nu(st)\,dt=\frac{\Gamma\left(\frac{\mu+\nu}{2}\right)}{2^{\nu+1}\Gamma(\nu+1)}\cdot\frac{s^\nu}{\rho^{\mu+\nu}}\cdot{}_1F_1\left({(\mu+\nu)/2 \atop \nu+1};-\frac{s^2}{4\rho^2}\right),\quad \mathrm{Re}(\mu+\nu)>0.$$

It can be shown that the integral converges for $\mathrm{Re}(\mu+\nu)>0$. Now we use the definition (1) to obtain

$$\int_0^\infty t^{\mu-1}e^{-\rho^2t^2}J_\nu(st)\,dt=\sum_{k=0}^\infty\frac{(-1)^k}{\Gamma(\nu+k+1)\,k!}\cdot\frac{s^{\nu+2k}}{2^{\nu+2k}}\cdot\int_0^\infty t^{\mu+\nu+2k-1}e^{-\rho^2t^2}\,dt.$$

Using the substitution $\rho^2t^2=u$ we find that

$$\begin{aligned}\int_0^\infty t^{\mu+\nu+2k-1}e^{-\rho^2t^2}\,dt &= \rho^{-\mu-\nu-2k+1}\int_0^\infty u^{(\mu+\nu+2k-1)/2}e^{-u}\frac{du}{2\rho\sqrt{u}}\\ &= \frac{1}{2}\rho^{-\mu-\nu-2k}\int_0^\infty u^{(\mu+\nu+2k-2)/2}e^{-u}\,du\\ &= \frac{1}{2}\rho^{-\mu-\nu-2k}\,\Gamma\left(\frac{\mu+\nu+2k}{2}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}\int_0^\infty t^{\mu-1}e^{-\rho^2t^2}J_\nu(st)\,dt &= \frac{1}{2}\rho^{-\mu-\nu}\sum_{k=0}^\infty\frac{(-1)^k\,\Gamma\left(\frac{\mu+\nu}{2}+k\right)}{\Gamma(\nu+k+1)\,k!}\cdot\frac{s^{\nu+2k}}{2^{\nu+2k}\,\rho^{2k}}\\ &= \frac{\Gamma\left(\frac{\mu+\nu}{2}\right)}{2\,\rho^{\mu+\nu}\,\Gamma(\nu+1)}\cdot\frac{s^\nu}{2^\nu}\cdot\sum_{k=0}^\infty\frac{(-1)^k((\mu+\nu)/2)_k}{(\nu+1)_k\,k!}\cdot\frac{s^{2k}}{2^{2k}\,\rho^{2k}}\\ &= \frac{\Gamma\left(\frac{\mu+\nu}{2}\right)}{2^{\nu+1}\Gamma(\nu+1)}\cdot\frac{s^\nu}{\rho^{\mu+\nu}}\cdot{}_1F_1\left({(\mu+\nu)/2 \atop \nu+1};-\frac{s^2}{4\rho^2}\right).\end{aligned}$$

The special case $\mu=\nu+2$ is of special interest: in that case we have $(\mu+\nu)/2=\nu+1$. This implies that the ${}_1F_1$ reduces to a ${}_0F_0$ which is an exponential function. The result is

$$\int_0^\infty t^{\nu+1}e^{-\rho^2t^2}J_\nu(st)\,dt=\frac{s^\nu}{(2\rho^2)^{\nu+1}}\cdot e^{-s^2/4\rho^2},\quad \mathrm{Re}\,\nu>-1.$$

Hankel functions

The functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are defined by

$$H_\nu^{(1)}(z) := J_\nu(z) + iY_\nu(z) \quad \text{and} \quad H_\nu^{(2)}(z) := J_\nu(z) - iY_\nu(z).$$

These functions are called Hankel functions or Bessel functions of the third kind.
Note that these definitions imply that

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2} \quad \text{and} \quad Y_\nu(z) = \frac{H_\nu^{(1)}(z) - H_\nu^{(2)}(z)}{2i}.$$

Further we have

$$H_{-1/2}^{(1)}(x) = J_{-1/2}(x) + iY_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,(\cos x + i \sin x) = \sqrt{\frac{2}{\pi x}}\,e^{ix}, \quad x > 0$$

and

$$H_{-1/2}^{(2)}(x) = J_{-1/2}(x) - iY_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,(\cos x - i \sin x) = \sqrt{\frac{2}{\pi x}}\,e^{-ix}, \quad x > 0.$$

Similarly we have

$$H_{1/2}^{(1)}(x) = J_{1/2}(x) + iY_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,(\sin x - i \cos x) = -i\sqrt{\frac{2}{\pi x}}\,e^{ix}, \quad x > 0$$

and

$$H_{1/2}^{(2)}(x) = J_{1/2}(x) - iY_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,(\sin x + i \cos x) = i\sqrt{\frac{2}{\pi x}}\,e^{-ix}, \quad x > 0.$$

Modified Bessel functions

The modified Bessel function $I_\nu(z)$ of the first kind of order ν is defined by

$$I_\nu(z) := \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix};\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{1}{\Gamma(\nu+k+1)\,k!}\left(\frac{z}{2}\right)^{2k}.$$

For $\nu \geq 0$ this is a solution of the modified Bessel differential equation

$$z^2 y''(z) + zy'(z) - \left(z^2 + \nu^2\right) y(z) = 0, \quad \nu \geq 0.$$

For $\nu \notin \{0, 1, 2, \ldots\}$ we have that $I_{-\nu}(z)$ is a second solution of this differential equation and the two solutions $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent.
For $\nu = n \in \{0, 1, 2, \ldots\}$ we have $I_{-n}(z) = I_n(z)$.
The modified Bessel function $K_\nu(z)$ of the second kind of order ν is defined by

$$K_\nu(z) := \frac{\pi}{2\sin\nu\pi}\,[I_{-\nu}(z) - I_\nu(z)] \quad \text{for} \quad \nu \notin \{0, 1, 2, \ldots\}$$

and

$$K_n(z) = \lim_{\nu \to n} K_\nu(z) \quad \text{for} \quad n \in \{0, 1, 2, \ldots\}.$$

Now we have for $x > 0$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sinh x, \quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cosh x \quad \text{and} \quad K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}.$$

A generating function

The Bessel function $J_n(z)$ of the first kind of integer order $n \in \mathbb{Z}$ can also be defined by means of the generating function

$$\exp\left(\frac{1}{2}z\left(t-t^{-1}\right)\right)=\sum_{n=-\infty}^{\infty}J_n(z)t^n. \tag{8}$$

In fact, the series on the right-hand side is a so-called Laurent series at $t=0$ for the function at the left-hand side. Using the Taylor series for the exponential function we obtain

$$\exp\left(\frac{1}{2}z\left(t-t^{-1}\right)\right)=\exp\left(\frac{zt}{2}\right)\cdot\exp\left(-\frac{z}{2t}\right)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{zt}{2}\right)^j\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2t}\right)^k=\sum_{n=-\infty}^{\infty}a_nt^n.$$

For $n \in \{0,1,2,\ldots\}$ we have

$$a_n=\sum_{k=0}^{\infty}\frac{1}{(n+k)!}\left(\frac{z}{2}\right)^{n+k}\cdot\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^k=\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)!\,k!}\left(\frac{z}{2}\right)^{2k}=J_n(z)$$

and

$$a_{-n}=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{z}{2}\right)^j\cdot\frac{(-1)^{j+n}}{(n+j)!}\left(\frac{z}{2}\right)^{n+j}=(-1)^nJ_n(z)=J_{-n}(z).$$

This proves (8).

If $t=e^{i\theta}$, then we have

$$\frac{1}{2}\left(t-t^{-1}\right)=\frac{e^{i\theta}-e^{-i\theta}}{2}=i\sin\theta$$
$$\Longrightarrow\quad\exp\left(\frac{1}{2}x\left(t-t^{-1}\right)\right)=e^{ix\sin\theta}=\cos(x\sin\theta)+i\sin(x\sin\theta).$$

Hence we have

$$\cos(x\sin\theta)+i\sin(x\sin\theta)=e^{ix\sin\theta}=\sum_{n=-\infty}^{\infty}J_n(x)e^{in\theta}=\sum_{n=-\infty}^{\infty}J_n(x)\left[\cos(n\theta)+i\sin(n\theta)\right].$$

Since $J_{-n}(x)=(-1)^nJ_n(x)$ this implies that

$$\cos(x\sin\theta)=\sum_{n=-\infty}^{\infty}J_n(x)\cos(n\theta)=J_0(x)+2\sum_{k=1}^{\infty}J_{2k}(x)\cos(2k\theta)$$

and

$$\sin(x\sin\theta)=\sum_{n=-\infty}^{\infty}J_n(x)\sin(n\theta)=2\sum_{k=0}^{\infty}J_{2k+1}(x)\sin(2k+1)\theta.$$

For $\theta=\pi/2$ this implies that

$$\cos x=J_0(x)+2\sum_{k=1}^{\infty}(-1)^kJ_{2k}(x)\quad\text{and}\quad\sin x=2\sum_{k=0}^{\infty}(-1)^kJ_{2k+1}(x).$$

For $\theta=0$ we also have

$$1=J_0(x)+2\sum_{k=1}^{\infty}J_{2k}(x).$$

The generating function (8) can be used to prove that

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y).$$

The proof is as follows:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} J_n(x+y)t^n &= \exp\left(\frac{1}{2}(x+y)\left(t-t^{-1}\right)\right) \\
&= \exp\left(\frac{1}{2}x\left(t-t^{-1}\right)\right) \cdot \exp\left(\frac{1}{2}y\left(t-t^{-1}\right)\right) \\
&= \sum_{k=-\infty}^{\infty} J_k(x)t^k \cdot \sum_{m=-\infty}^{\infty} J_m(y)t^m = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y)\right)t^n.
\end{aligned}$$

We can also find an integral representation for the Bessel function $J_n(x)$ of the first kind of integral order n starting from the generating function

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta}.$$

We use the orthogonality property of the exponential function, id est

$$\int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = \begin{cases} 0, & k \in \mathbb{Z}, \quad k \neq 0 \\ 2\pi, & k = 0. \end{cases}$$

Hence we have

$$\int_{-\pi}^{\pi} e^{ix\sin\theta} \cdot e^{-in\theta}\, d\theta = \sum_{k=-\infty}^{\infty} J_k(x) \int_{-\pi}^{\pi} e^{ik\theta} \cdot e^{-in\theta}\, d\theta = 2\pi \cdot J_n(x).$$

This implies that

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\theta - n\theta)}\, d\theta = \frac{1}{\pi}\int_0^{\pi} \cos\left(x\sin\theta - n\theta\right)\, d\theta.$$

The special case $n = 0$ reads

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos\left(x\sin\theta\right)\, d\theta = \frac{2}{\pi}\int_0^1 \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt.$$